Hadron masses in a chirally symmetric confining model

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Hadron masses are investigated in the framework of a chirally invariant, effective quark Hamiltonian. The hyperfine, spin-orbit, and tensor forces are found to be connected to the chiral angle, which measures the extent of vacuum condensation. The only finite eigenvalues of this Hamiltonian are shown to be color singlets. The chiral angle is obtained as a solution of the associated mass-gap equation. With the same bare parameters used when studying mesons with light quarks, we are able to account for the spectroscopy of charmonium and the N-Δ mass difference. The results are reasonably good.

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I. INTRODUCTION

Recently, in a series of papers [1–3], we have introduced a microscopic quark model where chiral symmetry and color confinement were consistently taken into account. Consistency was obtained by solving, for the chiral angle, the appropriate mass-gap equation of the model [2–5]. The chiral angle measures the extent of quark-antiquark condensate of the vacuum and was shown to affect the rate of mesonic decay and hence the resonance widths. In addition to the pion, we were able to describe the ρ and the ϕ resonances.

In those articles, it was also shown that bare meson wave functions, i.e., prior to decay, must have at least two components—the so-called positive- and negative-energy components. In the chiral limit of zero current quark masses, the pion behaves as a Goldstone boson and these two components are, to within a possible difference of a phase, degenerate. By turning on the quark current masses we lift this degeneracy, explicitly break chiral symmetry, and the pion acquires a small mass. Now, the ρ being a spin triplet, is not affected by the spontaneous chiral-symmetry-breaking mechanism. Its mass and wave function are essentially given by the Schrödinger-like one-channel equation. In other words, although the ρ is a bound state of light quarks, it contains only a small admixture of the "negative-energy" wave-function component. It is, unlike the π, a highly nondegenerate system.

This extra degree of freedom, called E spin, is unavoidable when dealing with bound states of fermions. Were it not for the E spin, the pion mass would be, for relatively light current quark masses, forbiddingly high and similar to the ρ mass. In fact, we would recover the old hyperfine splitting between the π and ρ.

This observation prompted us to see whether we could also describe, at the BCS level and with the same parameters and success as before, both the charmonium and the baryonic sector. If this proves successful, then we will have established a direct link between the chiral angle and, not only the strength of the hyperfine splitting, but also the strength of the spin-orbit and tensorial forces. This, while preserving the smallness of the pion mass, will allow the usual results of the quark model [6]. There is, however, one notable difference: In this model, and unlike other quark models, the strengths of the spin-spin, spin-orbit, and tensorial forces are connected through the same physics and are not separate.

At this point it should be stressed that the existence of these interactions and their global features do not depend on the detailed form of the chosen confining potential, provided it supports a π Goldstone boson. In fact it is not clear how to integrate the gluonic degrees of freedom, nor if, due to the non-Abelian nature of QCD, these gluons acquire a mass, in which case the instantaneous potential could give a good approximation [7]. Another source of theoretical uncertainty stems from the possible three-body nature of gluonic forces when dealing with the baryonic sector. It should be noticed that lattice gauge simulations seem to indicate that these three-body forces are small, and the microscopic quark interactions seem to be dominated by additive potential insertions [8].

Although we work with an instantaneous potential, it should be clear that the effect we are studying still holds for a proper covariant potential, as long as it admits vacuum quark condensation, i.e., spontaneous chiral-symmetry breaking. In addition, since we are using relativistic kinematics from the very beginning, this approach goes one step further in hadronic spectroscopy, as compared with nonrelativistic quark model calculations. However, it is true that for actual detailed predictions involving recoiling hadrons in the final state, one must use a covariant form for the microscopic potential. This entails extended and cumbersome calculations which we defer to a forthcoming article. To our knowledge, this has not been accomplished in a satisfactory way (an effort in this direction can be seen in Ref. [9]).

Many potential models and lattice calculation results for charmonium seem to favor a linear confining potential [10]. Although some degree of scalar potential admixture is still compatible with chiral symmetry breaking, and henceforth with a small mass for the pion, a pure scalar potential is not [1]. However, the evidence for a purely scalar confinement ansatz deduced from the behavior of the spin-orbit potential and extracted using nonchirally rotated spinors, from quenched lattice simulations seems, in our view, not absolutely compelling. Here we point out that the mere existence of the chiral angle, affecting the strength of the microscopic quark interaction (including the spin-orbit potential) changes the results and the new physical features arising from this fact alone, makes it worthwhile to pursue the study of the phenomenological implications of chiral condensation, using the harmonic potential as a first step. This choice allows us to turn the mass gap equation from a nonlinear integrodifferential equation into a differential equation. Work using a more realistic linear potential is in progress.

Following Ref. [1], we prefer to work in the formalism of quark and antiquark field operators $b^\dagger$ and $d^\dagger$ instead of the Feynman field operators ψ. In this equivalent formalism, we absorb the spinorial matrix elements $u_s(\mathbf{p})$ and $v_{s'}(\mathbf{p}')$ into the vertices, and keep the quark propagators at their simplest form. It happens that these (Valatin-Bogoliubov rotated) spinors contain, through the chiral angle, information about the vacuum condensate. Therefore, it should be no surprise that they yield effective chirally induced spin-spin, spin-orbit, and tensorial forces. The fact that these forces, weighted by appropriate functions of the chiral angle (which is a solution of the mass-gap equation), are in the range of the observed spectrum, for the bare parameters used in Refs. [1–3], constitutes *per se* an interesting result.

The paper is organized into six sections and two appendices. In Sec. II we briefly describe the model. In Sec. III, we show how this model leads to color confinement. Section IV is devoted to the study of mesons and baryons as bound states of quarks. The details of the derivation of bound-state equations are given in Appendix A. In Sec. V, the several terms arising from the q-q interaction and their relation to the chiral angle are studied. In this section, the bare masses of charmonium, nucleons, and the delta are evaluated solving the effective Schrödinger equation. In the meson case we use the Runge-Kutta method, while for baryons (and mesons with only one channel) we use the variational method. Appendix B contains the details of the variational method. We conclude in Sec. VI.

II. QUARK MODEL WITH CHIRAL-SYMMETRY BREAKING

The nonperturbative nature of QCD at low energies does not rule out, in principle, the possibility of constructing a quark model. For the same dynamics, the quark interaction can be defined in several ways: We could consider the microscopic interaction, for a given asymptotic system of n quarks and antiquarks, to be given by the sum of all n-irreducible Green's functions, i.e., Green's functions (truncated of the exterior $2n$ fermion legs) which cannot be cut in two, simply by cutting n interior fermion legs. Hadrons are then obtained by summing, in the *ladder* approximation, the kernel.

An alternative way of summing these Feynman diagrams is provided by the *resonating group method* [3]. There, the microscopic potential is given solely by those diagrams which do not contain fermion loops. Coupled channels are thereafter used to consistently evaluate the effect of those missed diagrams through the intermediate mesonic degrees of freedom. In this case, the interaction can be defined from the Green's functions of *pure gauge* QCD with 2,3,... gluon legs.

The quark model can be properly used only when the Green's function with two gluon legs is dominant with respect to the others. Only in this approximation is it possible to use the same two-body interaction to study the mass-gap equation (one fermion), mesons (two fermions), baryons (three fermions) and coupled channels which effectively include fermion loops.

This approximation has been tested in Refs. [1–3] using a microscopic quark model which leads to confinement and chiral-symmetry breaking in a consistent way. In this paper we extend that study to the case of heavy mesons and baryons.

First let us briefly review the Hamiltonian of the model which is

$$H = \int d^3x [H_0(\mathbf{x}) + H_I(\mathbf{x})] , \tag{1}$$

where H_0 is the Hamiltonian density of the Dirac field, and H_I an effective interaction term:

$$H_0(\mathbf{x}) = \psi^\dagger(\mathbf{x}) \; (m\beta - i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}) \; \psi(\mathbf{x}) , \tag{2}$$

$$H_I(\mathbf{x}) = \frac{1}{2}\int d^3y \, V(\mathbf{x}-\mathbf{y}) \, \psi^\dagger(\mathbf{x})\frac{\lambda^a}{2}\psi(\mathbf{x}) \times\psi^\dagger(\mathbf{y})\frac{\lambda^a}{2}\psi(\mathbf{y}) . \tag{3}$$

The λ^a's are the Gell-Mann color matrices. The spinor structure of this effective interaction is "Coulombic." In principle, there could be terms other than Coulombic, in which case the formalism we use here can be generalized to a wide range of combinations leading to similar mass-gap equations and chiral angles (see for example Refs. [1] and [9]).

The field operator has the form

$$\psi_{fc}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^{3/2}} [\, u_s(\mathbf{p}) b_{fsc}(\mathbf{p}) + v_s(\mathbf{p}) d^\dagger_{fsc}(-\mathbf{p})] e^{i\mathbf{p}\cdot\mathbf{x}} . \tag{4}$$

Here the Fock-space operators b and d refer to quark and antiquark, respectively, and they carry indices for flavor, spin, and color. Summation over repeated indices is used throughout this paper. The spinors u and v, and the Fock-space operators are not the same ones used in free Dirac theory, but rather linear combinations of them. Indeed, it amounts to choosing a specific Fock space which differs from the "naive" one. For u_s and v_s we have

$$u_s(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[f(p) + g(p)\hat{\mathbf{p}}\cdot\boldsymbol{\alpha}\right]u_s^0 ,$$
$$v_s(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[f(p) - g(p)\hat{\mathbf{p}}\cdot\boldsymbol{\alpha}\right]v_s^0 , \tag{5}$$
$$f(p) \equiv \sqrt{1+\sin\varphi(p)} ,$$
$$g(p) \equiv \sqrt{1-\sin\varphi(p)} ,$$

where u_s^0 and v_s^0 are the usual spinor eigenvectors of γ_0 corresponding to eigenvalues ± 1. The function $\varphi(p)$ is called the *chiral angle* and indexes the different Fock spaces compatible with the Pauli principle. This chiral angle has been studied in detail in Ref. [1]. Here we point out that in the limit of zero current quark mass and zero potential (which forces the chiral angle to be zero), expression (5) yields the massless Dirac spinors. The other trivial limit arises for very massive fermions with $\varphi = \pi/2$, for all momenta small enough. In this case and for this region of momentum, Eq. (5) becomes an identity.

In terms of the Fock-space operators, the Hamiltonian becomes

$$H = H_2 + H_4 , \tag{6}$$
$$H_2 = \int d^3k\, E(k)\left[b^\dagger_{fsc}(\mathbf{k})b_{fsc}(\mathbf{k}) + d^\dagger_{fsc}(\mathbf{k})d_{fsc}(\mathbf{k})\right] , \tag{7}$$
$$H_4 = \frac{1}{2}\int d^3p\, d^3k\, d^3q V(\mathbf{q})\left(\frac{\lambda^a_{c_1c_2}\lambda^a_{c_3c_4}}{4}\right)\sum_{j,l=1}^{4} :\Theta^j_{c_1c_2}(\mathbf{p},\mathbf{p}+\mathbf{q})\Theta^l_{c_3c_4}(\mathbf{k},\mathbf{k}-\mathbf{q}): . \tag{8}$$

In H_4, the ten different terms obtained when summing over the indices j and l, are combinations of the following vertices Θ^j:

$$\Theta^1_{c'c}(\mathbf{p},\mathbf{p}') \equiv u^\dagger_{s'}(\mathbf{p}')u_s(\mathbf{p})\, b^\dagger_{fs'c'}(\mathbf{p}')b_{fsc}(\mathbf{p}) ,$$
$$\Theta^2_{c'c}(\mathbf{p},\mathbf{p}') \equiv -v^\dagger_{s'}(\mathbf{p}')v_s(\mathbf{p})\, d^\dagger_{fsc}(-\mathbf{p})d_{fs'c'}(-\mathbf{p}') ,$$
$$\Theta^3_{c'c}(\mathbf{p},\mathbf{p}') \equiv u^\dagger_{s'}(\mathbf{p}')v_s(\mathbf{p})\, b^\dagger_{fs'c'}(\mathbf{p}')d^\dagger_{fsc}(-\mathbf{p}) , \tag{9}$$
$$\Theta^4_{c'c}(\mathbf{p},\mathbf{p}') \equiv v^\dagger_{s'}(\mathbf{p}')u_s(\mathbf{p})\, d_{fs'c'}(-\mathbf{p}')b_{fsc}(\mathbf{p}) .$$

The two terms H_2 and H_4 have been normal ordered. The normal ordering of the potential-energy operator introduces self-energy terms which are included in H_2, to give the quark energy $E(k)$:

$$E(k) = A(k)\sin\varphi(k) + B(k)\cos\varphi(k) , \tag{10}$$
$$A(k) \equiv m + \frac{2}{3}\int d^3p\, V(\mathbf{k}-\mathbf{p})\sin\varphi(p) , \tag{11}$$
$$B(k) \equiv k + \frac{2}{3}\int d^3p(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})V(\mathbf{k}-\mathbf{p})\cos\varphi(p) . \tag{12}$$

There are also constant terms arising from the normal ordering of both the kinetic- and the potential-energy operators. Their sum yields the energy of the vacuum condensate.

The Ward identity [11] with kernel H_4 gives, at least at the BCS level, the *mass-gap equation*.

$$A(k)\cos\varphi(k) - B(k)\sin\varphi(k) = 0 , \tag{13}$$

which defines the chiral angle $\varphi(k)$. This equation ensures that the vacuum condensate is stable: $\varphi(k)$ turns out to minimize the vacuum energy [5].

III. COLOR CONFINEMENT

In this paper, following Refs. [1–3] and [5], we choose to work with the potential

$$V(\mathbf{x}) = -K_0^3\mathbf{x}^2 + U. \tag{14}$$

The necessity of having asymptotic free quarks for defining "in" and "out" fields has forced us to introduce in the expression above a "constant" interquark potential shift U, independent of space coordinates. Both K_0 and U are positive and have dimensions of energy. Then, the total q-$\bar{q}$ potential can be seen as the limit, when $U \to +\infty$, of a succession of deeper and deeper potentials with eventually $V(\pm\infty) = 0$. Notice that U does not correspond to a universal, first-quantization, shift of the hadronic masses. It enters in the Hamiltonian (3) multiplied by a product of four fermion field operators ψ and two color matrices. Therefore, it will correspond to an operator and not to a c number. Furthermore, as we will see below, the effective interaction among quarks, in spite of V being positive, will be attractive in hadrons.

The function $V(\mathbf{q})$ in (8) has been defined as the Fourier transform of $V(\mathbf{x})$ divided by $(2\pi)^3$; then

$$V(\mathbf{q}) = K_0^3\nabla^2{}_q\delta(\mathbf{q}) + U\delta(\mathbf{q}). \tag{15}$$

We proceed now to prove a series of simple results.

(a) When $U\to +\infty$, the quark (antiquark) self-energy approaches plus infinity. This can be easily seen when considering the contributions of such a constant shift U to the functions $A(k)$ and $B(k)$ [see (11) and (12)]:

$$A_{[U]}(k) = \frac{2}{3}\int d^3p\; U\; \delta(\mathbf{k}-\mathbf{p})\sin\;\varphi(p) = \frac{2}{3}U\sin\varphi(k),$$
$$B_{[U]}(k) = \frac{2}{3}\int d^3p\;(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})U\;\delta(\mathbf{k}-\mathbf{p})\cos\;\varphi(p) \tag{16}$$
$$= \frac{2}{3}U\cos\varphi(k),$$

which gives $\frac{2}{3}U$ for the quark (antiquark) self-energy $E(k)$. The parts of $A(k)$ and $B(k)$ which depend on K_0 are well-behaved functions of $\mathbf{k}$, and independent of U[1–3]. If $U \to \infty$, then $E(k) = \frac{2}{3}U \;\to +\infty$ and therefore, we cannot have free quarks with finite energy.

(b) Such a shift of the interquark potential leaves the BCS mass-gap equation invariant. In fact the contribution of U to (13) is

$$\frac{2}{3} U \left(\cos\varphi(k)\sin\varphi(k) - \sin\varphi(k)\cos\varphi(k)\right) = 0. \quad (17)$$

(c) A "constant" potential does not yield quark-antiquark annihilation (creation) amplitudes. For a constant potential, the Feynman rules are simply given by

potential insertions:

$$\frac{\lambda \cdot \lambda}{4} U$$

vertices:

$$\begin{aligned} u_r^\dagger(k)\, u_s(k) &= \delta_{rs}, \\ -v_r^\dagger(k)\, v_s(k) &= -\delta_{rs}, \\ u_r^\dagger(k)\, v_s(k) &= 0, \\ v_r^\dagger(k)\, u_s(k) &= 0. \end{aligned} \quad (18)$$

It is clear that a constant potential does not yield q-$\overline{q}$ pairs.

(d) The Salpeter equation for mesons is invariant under the set of potential shifts parametrized by U. This is also very easy to see. Consider, for instance, the Salpeter equation for the positive-energy mesonic wave function [2,5]:

$$\begin{aligned} [M - 2E(k)]\phi^+_{s_1 s_2}(\mathbf{k}) = &-\frac{4}{3}\int d^3k'\, V(\mathbf{k}-\mathbf{k}')\, [u^\dagger_{s_1}(\mathbf{k})\, u_{s_3}(\mathbf{k}')]\, [v^\dagger_{s_4}(\mathbf{k}')v_{s_2}(\mathbf{k})]\phi^+_{s_3 s_4}(\mathbf{k}') \\ &+\frac{4}{3}\int d^3k'\, V(\mathbf{k}-\mathbf{k}')\, [u^\dagger_{s_1}(\mathbf{k})\, v_{s_4}(\mathbf{k}')]\, [u^\dagger_{s_3}(\mathbf{k}')v_{s_2}(\mathbf{k})]\phi^-_{s_3 s_4}(\mathbf{k}'). \end{aligned} \quad (19)$$

The contribution of U to the integral, which is $-\frac{4}{3}U$, cancels against the total shift of $2\left(\frac{2}{3}U\right)$ coming from the added self-energies of the quark and antiquark. This is referred to in the literature [5] as the infrared stability of the physical quantities. Notice also that a constant potential does not connect positive-energy wave-function components with negative-energy components.

We point out that the potential energy terms in (19) are *diagonal* in color because the following matrix, appearing in the binding energy of a meson, is diagonal:

$$\frac{1}{4}\lambda^a\, \lambda^a = \frac{4}{3}. \quad (20)$$

The minus sign in the quark-antiquark interaction inside a meson comes from the normal ordering of the antiquark vertex (antiquarks have negative color charge).

(e) This Hamiltonian confines color. Consider a system with a total number N of quarks and antiquarks. The contribution for the mass of such system is given by the sum of the quarks' (and antiquarks') kinetic energy plus the interquark potential energy. The "constant" potential contribution for the total energy is given by

$$\begin{aligned} &\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \frac{\lambda_i^a\lambda_j^a}{4}\, U = [\tfrac{1}{8}\Lambda^a\Lambda^a - \tfrac{2}{3}N]U, \\ &\Lambda^a \equiv \sum_{i=1}^{N}\lambda_i^a. \end{aligned} \quad (21)$$

In obtaining the result above we used (20). The total kinetic energy is, for the same "constant" potential, given by

$$E_{\rm kinetic} = \frac{2}{3}\, N\, U \quad (22)$$

which cancels against a similar term in Eq. (21), leaving for the total energy of a system composed of N quarks and antiquarks the simple expression

$$E_{\rm total} = \frac{1}{8}\Lambda^a\, \Lambda^a\, U. \quad (23)$$

Now, if we have color singlets, $\Lambda = 0$, and we have no contribution of such potentials. If, on the other hand, we have a colored ensemble of quarks and antiquarks, we have a positive contribution, proportional to $\Lambda^2 U$, for the mass of such a system.

When U goes to plus infinity, the mass of a colored object goes also to infinity. Only color singlets escape this fate; therefore, the only physical states are color singlets.

IV. BOUND-STATE EQUATIONS

Light-quark mesons were studied in Refs. [2] and [5], where it has been shown that the negative-energy component of the wave function was only relevant for pseudoscalar mesons which play the role of the Goldstone boson when $m = 0$. Whenever the quark or antiquark have a mass bigger than $(4/3)^{1/3}K_0$, the negative-energy component of the mesonic wave functions can be neglected. In the case of baryons, the negative-energy component does not exist because the two-body potential is clearly insufficient to annihilate a baryon-antibaryon pair.

Since our Hamiltonian (6) is instantaneous and we neglect negative-energy channels, the bound-state equation can be written in the form

$$H\,|\psi\rangle = M\,|\psi\rangle\,, \quad (24)$$

where $|\psi\rangle$ is an eigenstate of the Hamiltonian, with mass M.

Here we report on our results for bare baryons and charmed mesons, i.e., hadrons without coupled hadronic channels. Hence, we are not going to consider those vertices which create or annihilate q-$\overline{q}$ pairs. We postpone this study to a later paper.

A. Mesons as bound systems of two quarks

As discussed in the previous section, the only physical states, i.e., with a finite energy, are those which are color singlets. Therefore, we restrict our treatment to color singlets and positive-energy channels. Under these assumptions, the general form of the operator which creates a meson is given by

$$\Psi_m^\dagger = \int d^6p\, \delta(\mathbf{p}_1+\mathbf{p}_2)\, \psi(\mathbf{p}_1,\mathbf{p}_2) \times \chi_{f_1 f_2 s_1 s_2}\, b^\dagger_{f_1 s_1 c}(\mathbf{p}_1) d^\dagger_{f_2 s_2 c}(\mathbf{p}_2)\,. \tag{25}$$

The color indices of the two quarks have been contracted to obtain a color singlet. As a consequence, the wave function $\psi\chi$ can be taken to be symmetric under exchange of coordinates of the two quarks, without any loss of generality. Furthermore, for the ground state we will assume that the momentum part of the wave function, ψ, is symmetric.

The one-quark part of the Hamiltonian, Eq. (7), acting on the meson states gives

$$H_2\Psi_m^\dagger\,|0\rangle = \int d^6p\, \delta(\mathbf{p}_1+\mathbf{p}_2)\, [E(p_1)+\overline{E}(p_2)]\, \psi(\mathbf{p}_1,\mathbf{p}_2) \times \chi_{f_1 f_2 s_1 s_2}\, b^\dagger_{f_1 s_1 c}(\mathbf{p}_1) d^\dagger_{f_2 s_2 c}(\mathbf{p}_2)\,|0\rangle\,. \tag{26}$$

The kinetic energy $E(p)$ does not have the same functional form for all of the quarks involved, because in general their current masses can be different (different flavors) leading to different chiral angles. To emphasize that point we have used the notation $\overline{E}(p)$ for one of the quarks (antiquarks) involved in the microscopic scattering. The same is true of the spinors u and v which also depend on the chiral angle; consequently, we will use $\overline{v}$ for the second vertex in the potential.

Only one of the 10 terms in H_4 connects mesonic states with themselves, namely, the term

$$:\Theta^1_{c_1c_3}(\mathbf{p},\mathbf{p}')\overline{\Theta}^2_{c_2c_4}(\mathbf{k},\mathbf{k}') := -u^\dagger_{s_1}(\mathbf{p}')u_{s_3}(\mathbf{p})\overline{v}^\dagger_{s_4}(\mathbf{k}')\overline{v}_{s_2}(\mathbf{k}) b^\dagger_{f_1s_1c_1}(\mathbf{p}') d^\dagger_{f_2s_2c_2}(-\mathbf{k}) d_{f_2s_4c_4}(-\mathbf{k}') b_{f_1s_3c_3}(\mathbf{p})\,. \tag{27}$$

When the corresponding part of H_4 operates on the meson state the result is

$$H_4\Psi_m^\dagger\,|0\rangle = -\frac{4}{3}\chi_{f_1f_2s_3s_4}\int d^6p\, d^3q\, \delta(\mathbf{p}_1+\mathbf{p}_2)\, V(\mathbf{q})\, \psi(\mathbf{p}_1-\mathbf{q},\mathbf{p}_2+\mathbf{q}) \times u^\dagger_{s_1}(\mathbf{p}_1)\, u_{s_3}(\mathbf{p}_1-\mathbf{q})\, \overline{v}^\dagger_{s_4}(-\mathbf{p}_2-\mathbf{q})\, \overline{v}_{s_2}(-\mathbf{p}_2)\, b^\dagger_{f_1s_1c}(\mathbf{p}_1) d^\dagger_{f_2s_2c}(\mathbf{p}_2)\,|0\rangle\,. \tag{28}$$

The bound-state equation is obtained from (26) and (28):

$$[M-E(k)-\overline{E}(k)]\,\chi_{f_1f_2s_1s_2}\,\phi(\mathbf{k}) = -\frac{4}{3}\int d^3q\, V(\mathbf{q})\,[u^\dagger_{s_1}(\mathbf{k})u_{s_3}(\mathbf{k}-\mathbf{q})\overline{v}^\dagger_{s_4}(\mathbf{k}-\mathbf{q})\overline{v}_{s_2}(\mathbf{k})]\chi_{f_1f_2s_3s_4}\phi(\mathbf{k}-\mathbf{q})\,, \tag{29}$$

where $\phi((\mathbf{p}-\mathbf{p}')/2) \equiv \psi(\mathbf{p},\mathbf{p}')\,|_{\mathbf{p}+\mathbf{p}'=0}$. With the negative-energy channel excluded, this equation coincides with the Salpeter equation for mesons (19).

With the potential (15), the constant term U does not contribute, as explained in the previous section, while the harmonic term turns the bound-state equation into a second-order differential equation (a detailed derivation is given in Appendix A):

$$\Big[\frac{d^2}{dk^2} + M - E(k) - \overline{E}(k) - \frac{L(L+1)}{k^2} - \frac{\varphi'^2(k)+\overline{\varphi}'^2(k)}{4} + \frac{\sin\varphi(k)\,\sin\overline{\varphi}(k)-1}{k^2} + \frac{2}{k^2}[g^2(k)\,\mathbf{S}_1 + \overline{g}^2(k)\,\mathbf{S}_2]\cdot\mathbf{L} - \frac{2\,g^2(k)\,\overline{g}^2(k)}{k^2}\left(\frac{S}{3}(S+1) + (\hat{\mathbf{k}}\cdot\mathbf{S}_1)(\hat{\mathbf{k}}\cdot\mathbf{S}_2) - \frac{1}{3}\,\mathbf{S}_1\cdot\mathbf{S}_2\right)\Big]\nu(k) = 0\,. \tag{30}$$

In this last equation all momenta, energy and masses are given in units of $(4/3)^{1/3}K_0$. The operator $\mathbf{L}$ is the total angular momentum, $\mathbf{S}_i$ is the spin of the ith quark, and S is the total spin. This equation is to be solved by the numerical method used in Ref. [2].

B. Baryons

The formalism used to derive the bound-state equation for mesons can be easily generalized to bound states of three quarks. As mentioned at the beginning of the section, the wave functions of baryons have no contribution from negative-energy channels. Furthermore, in this paper we will restrict our treatment to baryons with quarks of the same current mass, leading to only one chiral angle φ, the same dispersion relation $E(p)$ for the three quarks, and only one set of spinors u and v in the vertices Θ. This constitutes a reasonable approximation in the case of the nucleon and Δ. We will also restrict the baryonic wave function to the simplest S-wave configuration, and discard, in what this paper is concerned, coupled channels.

As was done in the mesonic sector, the creation operator for baryons has the form

$$\Psi_b^\dagger = \int d^9p\, \delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\, \psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\, \epsilon_{c_1c_2c_3}\, \chi_{f_1f_2f_3s_1s_2s_3} b^\dagger_{f_1s_1c_1}(\mathbf{p}_1) b^\dagger_{f_2s_2c_2}(\mathbf{p}_2) b^\dagger_{f_3s_3c_3}(\mathbf{p}_3)\,. \tag{31}$$

The anticommutation relations for the quark operators ensure antisymmetry as required by Pauli exclusion. However, since the color part of the wave function, $\epsilon_{c_1c_2c_3}$, is completely antisymmetric, the flavor-spin-momentum wave function $\chi\psi$ can be taken to be symmetric.

The one-quark part of the Hamiltonian acting on the baryon state gives

$$H_2\Psi_b^\dagger\,|0\rangle = \int d^9p\,\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\,[E(p_1)+E(p_2)+E(p_3)]\,\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) \times\epsilon_{c_1c_2c_3}\chi_{f_1f_2f_3s_1s_2s_3}b^\dagger_{f_1s_1c_1}(\mathbf{p}_1)b^\dagger_{f_2s_2c_2}(\mathbf{p}_2)b^\dagger_{f_3s_3c_3}(\mathbf{p}_3)\,|0\rangle\,. \tag{32}$$

Without coupled channels, we have only to consider the following vertex product in the interaction:

$$:\Theta^1_{c_1c_4}(\mathbf{p},\mathbf{p}')\Theta^1_{c_2c_5}(\mathbf{k},\mathbf{k}') := u^\dagger_{s_1}(\mathbf{p}')u_{s_3}(\mathbf{p})u^\dagger_{s_2}(\mathbf{k}')u_{s_4}(\mathbf{k})b^\dagger_{f_1s_1c_1}(\mathbf{p}')b^\dagger_{f_2s_2c_2}(\mathbf{k}')b_{f_2s_4c_5}(\mathbf{k})b_{f_1s_3c_4}(\mathbf{p})\,, \tag{33}$$

which leads to

$$H_4\Psi_b^\dagger\,|0\rangle = -2\epsilon_{c_1c_2c_3}\chi_{f_1f_2f_3s_3s_4s_5}\int d^9p\,d^3q\;\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\;V(\mathbf{q})\psi(\mathbf{p}_1-\mathbf{q},\mathbf{p}_2+\mathbf{q},\mathbf{p}_3) \times[u^\dagger_{s_1}(\mathbf{p}_1)u_{s_3}(\mathbf{p}_1-\mathbf{q})u^\dagger_{s_2}(\mathbf{p}_2)u_{s_4}(\mathbf{p}_2+\mathbf{q})]b^\dagger_{f_1s_1c_1}(\mathbf{p}_1)b^\dagger_{f_2s_2c_2}(\mathbf{p}_2)b^\dagger_{f_3s_5c_3}(\mathbf{p}_3)\,|0\rangle\,. \tag{34}$$

Notice the factor of -2 which includes a contribution of $-2/3$ from the contraction of the color matrices times the number of quark pairs.

The resulting bound-state equation for baryons is

$$[M-3E(p_1)]\;\chi_{s_1s_2s_5}\;\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = -2\int d^3q\;V(\mathbf{q})\;[u^\dagger_{s_1}(\mathbf{p}_1)u_{s_3}(\mathbf{p}_1-\mathbf{q})u^\dagger_{s_2}(\mathbf{p}_2)u_{s_4}(\mathbf{p}_2+q)]\chi_{s_3s_4s_5}\psi(\mathbf{p}_1-\mathbf{q},\mathbf{p}_2+\mathbf{q},\mathbf{p}_3)\,, \tag{35}$$

with $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3 = 0$, for any flavor indices in the function χ. As in the case of mesons, Eq. (29), the harmonic potential (15) leads to a second-order differential equation, as shown in Appendix A:

$$\left\{3E(p_1) - M - \frac{3}{2}\nabla^2_{\mathbf{p}_{12}} + \frac{3}{4}\varphi'^2(p_1) + \frac{3[1-\sin\varphi(p_1)]}{p_1^2} + \left[\frac{3}{4} - \frac{1}{3}S(S+1)\right][1-\sin\varphi(p_1)][1-\sin\varphi(p_2)]\frac{\widehat{\mathbf{p}}_1\cdot\widehat{\mathbf{p}}_2}{p_1p_2}\right\}\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = 0\,, \tag{36}$$

where S is the total spin, and $\nabla^2_{\mathbf{p}_{12}}$ stands for the Laplacian with respect to the relative momentum $(\mathbf{p}_1-\mathbf{p}_2)/2$. We have omitted the spin-orbit and tensorial terms which vanish in the N-Δ case with only one channel. The larger number of variables in this equation makes it intractable with the numerical method used for mesons. Instead, we have used a variational method (Appendix B).

V. CHIRALLY INDUCED HADRON MASS SPECTRUM

The bound-state equations (30) and (36) resemble a Schrödinger equation with spin-spin, spin-orbit, and tensor interactions. As usual, these interactions will yield different masses for different mesons, depending upon total S, L, and J. However, all of these interactions have been derived from a *single* potential term of the Hamiltonian, and contain the "same" information of the chiral angle.

For mesons,

$$-\frac{4}{3}\int d^3q\;V(\mathbf{q})\;[u^\dagger_{s_1}(\mathbf{k})u_{s_3}(\mathbf{k}-\mathbf{q})\overline{v}^\dagger_{s_4}(\mathbf{k}-\mathbf{q})\overline{v}_{s_2}(\mathbf{k})]\cdots\,. \tag{37}$$

For baryons,

$$-2\int d^3q\;V(\mathbf{q})\;[u^\dagger_{s_1}(\mathbf{p}_1)u_{s_3}(\mathbf{p}_1-\mathbf{q})u^\dagger_{s_1}(\mathbf{p}_2)u_{s_3}(\mathbf{p}_2+\mathbf{q})]\cdots\,. \tag{38}$$

In what follows, we will consider each one of these interactions in detail.

A. Hyperfine splitting in light mesons

The terms proportional to S^2+S in Eq. (30) were derived from that part of the interaction which contains the operator $\mathbf{S}_1\cdot\mathbf{S}_2$. They stand for the effect, in the mesonic "positive-energy" channels, of the hyperfine interactions *weighted* by an appropriate function of the chiral angle:

$$-\frac{2}{3k^2}[1-\sin\varphi(k)][1-\sin\overline{\varphi}(k)]\,. \tag{39}$$

This hyperfine interaction plays its traditional role in separating the $^4S_{3/2}$ and $^2S_{1/2}$ baryon masses (Δ-nucleon mass difference). This is clearly seen in our results for the Δ-N mass difference (Table I).

But in the case of light mesons, in addition to this traditional contribution, the hyperfine term also enters the potential connecting "positive-energy" and "negative-energy" channels. This transition potential comes from the vertex structure of the mesonic Salpeter equation, which contains a term

$$[u^\dagger(\mathbf{k})\nabla v(\mathbf{k})]\cdot[\nabla u^\dagger(\mathbf{k})v(\mathbf{k})] \approx (\mathbf{k}\cdot\mathbf{S}_1)\times(\mathbf{k}\cdot\mathbf{S}_2)\,, \tag{40}$$

which yield both $\mathbf{S}_1\cdot\mathbf{S}_2$ and tensor terms.

Because of the large negative-energy wave-function component of the pion it turns out to be the mechanism *responsible* for the anomalous mass difference between the π and the ρ. It is instructive to see how this happens. If we neglect the $L=2$ wave-function component for the ρ, which has a small effect in the ρ mass, we have for the Salpeter equation [2] the following.

TABLE I. The masses of the nucleon and the Δ, for different current quark masses $m \equiv m_u \equiv m_d$. The potential strength is the same used for charmonium $(4/3)^{1/3}K_0 = 290$ MeV. The length α is the variational parameter (Appendix B).

m (MeV)	M_N (MeV)	M_Δ (MeV)	α_N (fm)	α_Δ (fm)
0	1378	1612	0.629	0.540
0.725	1378	1611	0.628	0.539
7.25	1375	1607	0.622	0.537
290	1844	2005	0.479	0.435

For the ρ,

$$\left[\left(\frac{d^2}{dk^2}-2E-\frac{8\,g^2(k)}{3k^2}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}+M\begin{pmatrix}1&0\\0&-1\end{pmatrix}-\frac{\varphi'^2(k)}{6}\begin{pmatrix}3&1\\1&3\end{pmatrix}+\frac{\cos^2\varphi(k)}{3k^2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\right]\begin{pmatrix}\nu_0^+\\\nu_0^-\end{pmatrix}=0\,. \tag{41}$$

For the π,

$$\left[\left(\frac{d^2}{dk^2}-2E\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}+M\begin{pmatrix}1&0\\0&-1\end{pmatrix}-\left(\frac{\varphi'^2(k)}{2}+\frac{\cos^2\varphi(k)}{k^2}\right)\begin{pmatrix}1&1\\1&1\end{pmatrix}\right]\begin{pmatrix}\nu_0^+\\\nu_0^-\end{pmatrix}=0\,. \tag{42}$$

It can be seen by direct inspection that the transition potential between these two energy channels goes as follows.

For the π,

$$\frac{\varphi'^2(k)}{2}+\frac{\cos^2\varphi(k)}{k^2}\,,$$

and for the ρ, (43)

$$\frac{1}{3}\left(\frac{\varphi'^2(k)}{2}+\frac{\cos^2\varphi(k)}{k^2}\right)\,.$$

In the ρ case, the transition potential turns out to be three times *smaller* than the corresponding potential of the π.

As a result, the hyperfine origin of the ρ-π mass difference is established, with the surprising result that it is realized in a generalized space which includes the so-called energy-spin space. This result does not depend on the actual form of the microscopic potential provided it supports spontaneous chiral-symmetry breaking, which in turn forces the pion to be a pseudo Goldstone boson (a true Goldstone boson in the chiral limit). The ρ being a vector escapes this fate. In fact, the ρ mass depends on the strength of the confining potential whereas, in the chiral limit, the π "does not," protected as it is by the chiral symmetry. The dynamical translation of this symmetry requirement forces a large hyperfine strength in the coupling of positive- to negative-energy spin pion wave-function components. Had we chosen a different potential also supporting dynamical chiral-symmetry breaking, the same mechanism would be bound to occur to keep the π at its smallest possible mass, compatible with the explicit, chiral-symmetry-breaking, quark current mass.

B. Spin-spin, spin-orbit, and tensor forces

In Appendix A, the effective spin-orbit and tensorial forces are derived. For completeness, in Table II we give

TABLE II. Spin-spin, spin-orbit, and tensorial forces in terms of φ.

Force	Mesons	Baryons
Spin-spin	$-\frac{4}{3k^2}g^2(k)\,\overline{g}^2(k)\,\mathbf{S}_1\cdot\mathbf{S}_2$	$\frac{\widehat{\mathbf{p}}_1\cdot\widehat{\mathbf{p}}_2}{3p_1p_2}g^2(p_1)\,g^2(p_2)\,\mathbf{S}_1\cdot\mathbf{S}_2$
Spin-orbit	$\frac{2}{k^2}\left[g^2(k)\,\mathbf{S}_1+\overline{g}^2(k)\,\mathbf{S}_2\right]\cdot\mathbf{L}$	$-\frac{6g^2(p_1)}{p_1^2}\mathbf{S}_1\cdot\mathbf{L}_1+\frac{3g^2(p_1)}{p_1^2}(\mathbf{S}_1+\mathbf{S}_2)\cdot(i\nabla_{p_2}\times\mathbf{p}_1)$
Tensor	$\frac{2}{k^2}g^2(k)\,\overline{g}^2(k)$ $[(\widehat{\mathbf{k}}\cdot\mathbf{S}_1)(\widehat{\mathbf{k}}\cdot\mathbf{S}_2)-\frac{1}{3}\mathbf{S}_1\cdot\mathbf{S}_2]$	$\frac{3\widehat{\mathbf{p}}_1\cdot\widehat{\mathbf{p}}_2}{p_1p_2}g^2(p_1)\,g^2(p_2)$ $\times[(\widehat{\mathbf{p}}_1\cdot\mathbf{S}_1)(\widehat{\mathbf{p}}_2\cdot\mathbf{S}_2)-\frac{1}{3}(\widehat{\mathbf{p}}_1\cdot\widehat{\mathbf{p}}_2)(\mathbf{S}_1\cdot\mathbf{S}_2)]$

their explicit form in terms of the chiral angle φ.

This spin-orbit interaction can be decomposed as the sum of normal and "anomalous" spin-orbit terms

$$\frac{2}{k^2}\left[g^2(k)\,\mathbf{S}_1+\overline{g}^2(k)\,\mathbf{S}_2\right]\cdot\mathbf{L}$$

$$=\frac{g^2(k)+\overline{g}^2(k)}{k^2}\mathbf{S}\cdot\mathbf{L}+\frac{g^2(k)-\overline{g}^2(k)}{k^2}(\mathbf{S}_1-\mathbf{S}_2)\cdot\mathbf{L}\,. \quad (44)$$

The "anomalous" term only appears for unequal quark masses and mixes the charge-conjugation eigenstates 1^{++} and 1^{+-}.

As the current quark mass m increases, the chiral angle $\varphi(k)$ approaches the limit $\pi/2$ (see Fig. 1), and all of these forces become zero, turning the bound-state equation into a Schrödinger equation with a harmonic-oscillator potential. Another interesting limit is when chiral symmetry is restored ($m \to 0$ and a potential which does not break chiral symmetry), which gives $\varphi \equiv 0$ and the function $g^2(k)=[1-\sin\varphi(k)]$ reaches its maximum.

In charmonium, the spin-orbit force is responsible for the coupling between $^3P_1$ and $^1P_1$ channels, and the tensorial force is responsible for the coupling of $^3S_1$ with $^3D_1$. For baryons, since we restrict our treatment to S-wave ground state without coupling to other angular momenta, the spin-orbit and tensorial terms do not appear in the bound-state equation. These various terms can now be used to obtain the charmonium mass spectrum, as well as the masses for the nucleon and Δ.

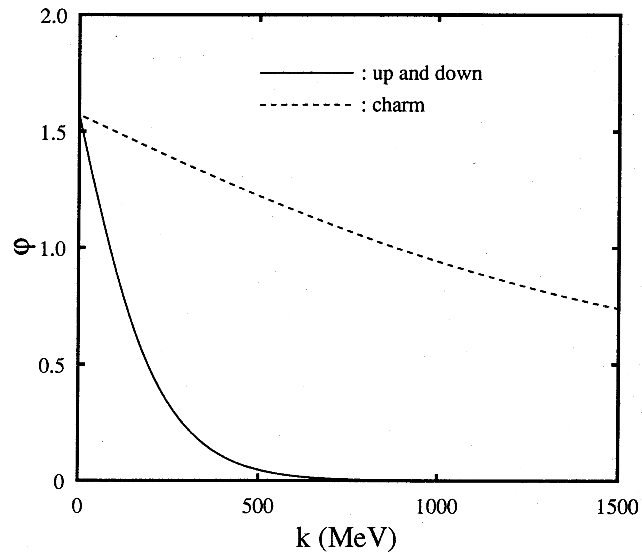

FIG. 1. The chiral angle for up and down quarks ($m_u = m_d = 0$), and charm ($m_c = 1362$ MeV). The potential is harmonic with a strength constant $(4/3)^{1/3}K_0 = 290$ MeV.

C. Charmonium

For charmonium we have the following cases (see Table III).

$^1S_0$: (0^-) and (0^{-+})

$$\left[\frac{d^2}{dk^2}-E(k)-\overline{E}(k)-\frac{\varphi'^2(k)+\overline{\varphi}'^2(k)}{4}-\frac{1-\sin\varphi(k)\,\sin\overline{\varphi}(k)}{k^2}+M\right]\nu=0\,. \quad (45)$$

$^3P_0$: (0^+) and (0^{++})

$$\left[\frac{d^2}{dk^2}-E(k)-\overline{E}(k)-\frac{2}{k^2}-\frac{\varphi'^2(k)+\overline{\varphi}'^2(k)}{4}-\frac{1-\sin\varphi(k)\sin\overline{\varphi}(k)}{k^2}+M\right]\nu=0\,. \quad (46)$$

$^3P_1\ ^1P_1$: (1^+) ; (1^{++}) and (1^{+-})

$$\left[\left(\frac{d^2}{dk^2}-E(k)-\overline{E}(k)-\frac{2}{k^2}-\frac{\varphi'^2(k)+\overline{\varphi}'^2(k)}{4}+M\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}\right.$$
$$\left.-\frac{1-\sin\varphi(k)\sin\overline{\varphi}(k)}{k^2}\begin{pmatrix}0&0\\0&1\end{pmatrix}+\frac{g^2(k)-\overline{g}^2(k)}{k^2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\begin{pmatrix}{}^3\nu\\{}^1\nu\end{pmatrix}=0\,. \quad (47)$$

$^3S_1,\ ^3D_1$: (1^-) ; (1^{--})

$$\left[\left(\frac{d^2}{dk^2}-E(k)-\overline{E}(k)-\frac{\varphi'^2(k)+\overline{\varphi}'^2(k)}{4}+M\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}-\frac{1}{k^2}\begin{pmatrix}6&0\\0&0\end{pmatrix}\right.$$
$$\left.+\frac{1-\sin\varphi(k)\sin\overline{\varphi}(k)}{3k^2}\begin{pmatrix}2&\sqrt{2}\\\sqrt{2}&1\end{pmatrix}-\frac{g^2(k)+\overline{g}^2(k)}{3k^2}\begin{pmatrix}-4&\sqrt{2}\\\sqrt{2}&4\end{pmatrix}\right]\begin{pmatrix}\nu_2\\\nu_0\end{pmatrix}=0\,. \quad (48)$$

$^3P_2\,^3F_2:\quad (2^+)\;;\;(2^{++})$

$$\left[\left(\frac{d^2}{dk^2}-E(k)-\overline{E}(k)-\frac{\varphi'^2(k)+\overline{\varphi}'^2(k)}{4}+M\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}-\frac{2}{k^2}\begin{pmatrix}6&0\\0&1\end{pmatrix}\right.$$

$$\left.+\frac{1-\sin\varphi(k)\sin\overline{\varphi}(k)}{5k^2}\begin{pmatrix}3&\sqrt{6}\\\sqrt{6}&2\end{pmatrix}-\frac{g^2(k)+\overline{g}^2(k)}{5k^2}\begin{pmatrix}-12&\sqrt{6}\\\sqrt{6}&12\end{pmatrix}\right]\begin{pmatrix}\nu_3\\\nu_1\end{pmatrix}=0\,. \quad (49)$$

We have solved these equations using the Runge-Kutta method and the numerical results are summarized in Table III.

D. Nucleons and Δ

The bound-state equation (36) for nucleons and the Δ has been solved using the variational method (Appendix B). As a test for the variational method we have used it in charmonium (for those cases with only one channel) giving good agreement with the results in Table III. Figure 2 shows the results for the N-Δ masses as a function of the variational parameter α, for the case of massless current quarks. The numerical results for several values of the current quark mass are given in Table I. Notice that for reasonably small values of m the results are, within numerical errors, identical to those of $m = 0$.

Regarding the relationship between the π-ρ and N-Δ mass differences and in order to complete the discussion at the end of Sec. V A, it is important to notice that baryons do not possess negative-energy spin wave-function components. Therefore, in this sector, the hyperfine interaction can only act, in contrast with light mesons, between positive-energy spin wave functions. This is why we can accommodate, with any prescribed microscopic potential supporting chiral-symmetry breaking, the low mass of the pion together with the "usual" results for baryonic spectroscopy.

The results for the N-Δ masses (Table I)are reasonably good, considering we have not included coupled channels. Coupling to mesons plays an important role in baryons; namely, a physical nucleon is surrounded by a cloud of virtual mesons. Consequently, it is expected that the bare masses in Table I move down significantly when coupled channels are included. In this context and although we lack a detailed calculation, our model possesses all the features of a genuine coupled-channel equation [3]. Therefore we adhere to the conclusions of the model-independent analysis of Thomas and Miller [13]: namely, the N and the Δ masses will move down around 300 MeV, the proportion of the N-Δ mass difference due to pion coupling ranges between 100 and 200 MeV, and it is erroneous to expect a larger Δ mass shift, due to pion coupling, than that of the nucleon.

So, it is to be expected that coupling to pion channels will improve the bare Δ-N result of 234 MeV. Finally

TABLE III. Mesonic spectrum in the charm sector, with $(4/3)^{1/3}K_0 = 290$ MeV and $m_c = 1362$ MeV. $m_u = m_d = 0$. The experimental values are from Ref. [12].

Meson	J^{PC}	^SL_J	Theory (MeV)	Experiment (MeV)
η_c	0^{-+}	$^1S_0$	3096	2979
J/ψ	1^{--}	$^3S_1+^3D_1$ [a]	3097	3097
χ_{c_0}	0^{++}	$^3P_0$	3332	3415
χ_{c_1}	1^{++}	$^3P_1$	3343	3511
χ_{c_2}	2^{++}	$^3P_2+^3F_2$ [a]	3365	3556
ψ'	1^{--}	$^3S_1+^3D_1$ [a]	3579	3686
ψ''	1^{--}	$^3S_1+^3D_1$ [a]	3611	3770
ψ'''	1^{--}	$^3S_1+^3D_1$ [a]	4155	4040
ψ''''	1^{--}	$^3S_1+^3D_1$ [a]	4209	4159
ψ'''''	1^{--}	$^3S_1+^3D_1$ [a]	4935	4415
D	0^-	$^1S_0$	1998	1869
D_0^*	0^+	$^3P_0$	2216	
D^*	1^-	$^3S_1$	2005	2007
D_1	1^+	$^3P_1+^1P_1$ [b]	2271	
D_1	1^+	$^3P_1+^1P_1$ [b]	2499	2424
D_2^*	2^+	$^3P_2+^3F_2$ [a]	2552	2459

[a]Tensorial coupling.
[b]Spin-orbit coupling.

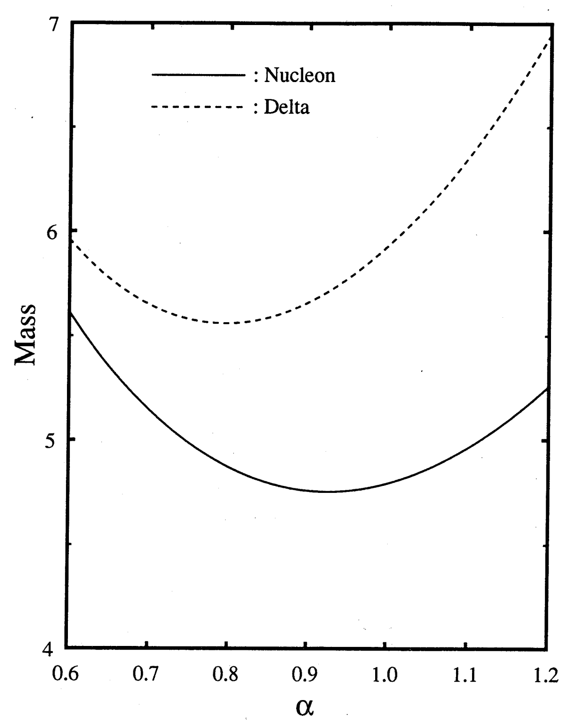

FIG. 2. Results of the variational method for the masses of the nucleon and the Δ. The mass M and the variational parameter α are in dimensionless units as described in the text.

we would like to stress that with only one parameter K_0, which determined the chiral angle φ, not only was the N-Δ hyperfine splitting reproduced, but also their bare radii (Table I). These will increase as a result of coupled channels.

VI. CONCLUSIONS

In this work we were able to obtain, at the BCS level and with some degree of success, the charmonium mass spectrum together with reasonable values for the nucleon and Δ masses. In addition to color confinement, this model predicts, for hadronic physics, a unified origin of the hyperfine, spin-orbit and tensor forces. All of these interactions stem from products of Valatin-Bogoliubov rotated spinors and hence contain, through the chiral angle, the information on the extent of the vacuum chiral condensation.

The hyperfine interaction was shown to account for the π-ρ mass difference provided we have an extra degree of freedom, called E spin. This degree of freedom comes naturally when using Dirac spinors in the framework of the Salpeter equation. For mesons other than the pseudoscalars, and especially for baryons, the negative-energy wave-function component is negligible or nonexistent and the physics involved becomes Schrödinger-like, with interaction strengths given by appropriate functions of the chiral angle.

Our results for the masses of mesons with charmed quarks follow the same ordering as the experimental ones. Our theoretical mass shifts are dominated by radial and spin-orbit splittings. Radial and angular excitations distort as expected the spectrum and suggest that a linear potential (possibly together with a Coulomb potential) would yield good results. Spin-orbit results seem to be of the correct order. The chiral angle correction is fundamental to understanding the D mesons.

That this vacuum condensation strongly affects the mass results, it is sufficient to see for instance that if in charmonium we had set the light-quark chiral angle to zero, we would have parity doublets which are not observed in nature, very small angular excitations, and the masses for D mesons lowered by hundreds of MeV's.

At this point it should be emphasized that we had no control on this chiral angle which is not a parameter but the solution of the associated (nonlinear) mass-gap equation.

Another point of interest lies in the spin-orbit force. It was shown to contain, for unequal $q\overline{q}$ masses, an "anomalous" term which is responsible for the mixing between the 1^{++} and 1^{+-} states causing the D mesons not to have a definite charge conjugation.

Despite the obvious need to incorporate coupled channels we were able to accommodate the N-Δ and the π-ρ mass differences with the same hyperfine microscopic interaction. This fact alone allows us to hint that, had we used this effective hyperfine potential, as deduced from our model, the interplay between Pauli principle and this hyperfine force would have led, along the lines of Ref. [14] (which linked the Δ-N mass difference to the N-N s-wave repulsion) also to a sizable N-N repulsion.

It will be interesting to verify if the obtained chirally induced hyperfine force is strong enough to produce a sizable N-N repulsive force.

Finally, the relative success of these results and the qualitative new effects it suggests should prompt further research using a more realistic covariant potential, presumably linear, together with post-BCS (coupled-channel) physics. This is a large program of which this work constitutes just a beginning.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix we derive the mesonic and baryonic bound-state equations for the potential (15). In the case of mesons, the general form of the bound states was given in (29); with the potential (15) it becomes

$$[M - E(k) - \overline{E}(k)]\, \chi_{f_1 f_2 s_1 s_2}\, \phi(\mathbf{k}) = -\frac{4K_0^3}{3} A_{s_1 s_2 s_3 s_4}(\mathbf{k})\, \chi_{f_1 f_2 s_3 s_4}\,, \quad \text{(A1)}$$

with

$$A_{s_1 s_2 s_3 s_4}(\mathbf{k}) \equiv \nabla_{\mathbf{q}}^2 [\, \phi(\mathbf{k}-\mathbf{q}) u_{s_1}^\dagger(\mathbf{k}) u_{s_3}(\mathbf{k}-\mathbf{q}) \times \overline{v}_{s_4}^\dagger(\mathbf{k}-\mathbf{q}) \overline{v}_{s_2}(\mathbf{k})]_{\mathbf{q}=0}\,. \quad \text{(A2)}$$

From the definition of the spinors u_s and v_s given in Sec. II, we see that

$$\begin{aligned} u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}') &= v_s^\dagger(\mathbf{p}') v_r(\mathbf{p})\,, \\ u_r^\dagger(\mathbf{p}) v_s(\mathbf{p}') &= v_r^\dagger(\mathbf{p}) u_s(\mathbf{p}') = 0\,, \\ u_r^\dagger(\mathbf{p}) u_s(\mathbf{p}) &= v_r^\dagger(\mathbf{p}) v_s(\mathbf{p}) = \delta_{rs}\,, \end{aligned} \quad \text{(A3)}$$

which leads to

$$\begin{aligned} A_{s_1 s_2 s_3 s_4}(\mathbf{k}) = \{ & \delta_{s_1 s_3}\delta_{s_2 s_4}\nabla^2 + \delta_{s_2 s_4} u_{s_1}^\dagger(\mathbf{k})\nabla^2 u_{s_3}(\mathbf{k}) + \delta_{s_1 s_3}\overline{u}_{s_2}^\dagger(\mathbf{k})\nabla^2 \overline{u}_{s_4}(\mathbf{k}) \\ & + 2[\delta_{s_2 s_4} u_{s_1}^\dagger(\mathbf{k})\boldsymbol{\nabla} u_{s_3}(\mathbf{k}) + \delta_{s_1 s_3}\overline{u}_{s_2}^\dagger(\mathbf{k})\boldsymbol{\nabla}\overline{u}_{s_4}(\mathbf{k})]\cdot\boldsymbol{\nabla} + 2u_{s_1}^\dagger(\mathbf{k})\boldsymbol{\nabla} u_{s_3}(\mathbf{k})\cdot\overline{u}_{s_2}^\dagger(\mathbf{k})\boldsymbol{\nabla}\overline{u}_{s_4}(\mathbf{k})\}\phi(\mathbf{k})\,. \end{aligned} \quad \text{(A4)}$$

The gradient and the Laplacian of the spinor u_r are

$$\boldsymbol{\nabla} u_r(\mathbf{p}) = \left[f'\hat{\mathbf{p}} + \frac{g}{p}\boldsymbol{\alpha} + \left(g' - \frac{g}{p}\right)(\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}} \right] u_r^0\,, \quad \text{(A5)}$$

$$\nabla^2 u_r(\mathbf{p}) = \left[f'' + \frac{2f'}{p} + \left(g'' + \frac{2g'}{p} - \frac{2g}{p^2} \right)\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} \right] u_r^0\,, \quad \text{(A6)}$$

and their product with the Hermitian-conjugate spinor $u^\dagger$ gives

$$u_r^\dagger(\mathbf{p})\nabla u_s(\mathbf{p}) = \frac{ig^2(p)}{2p}\,\widehat{\mathbf{p}}\times\boldsymbol{\sigma}_{rs}\,,$$

$$u_r^\dagger(\mathbf{p})\nabla^2 u_s(\mathbf{p}) = -\delta_{rs}\left(\frac{1}{4}\varphi'^2(p)+\frac{g^2(p)}{p^2}\right)\,. \tag{A7}$$

The notation $\boldsymbol{\sigma}_{rs}$ stands for the vector whose three coordinates are the (r,s) elements of the three Pauli matrices.

We have made use of the relations

$$ff'+gg'=0\,,$$
$$ff''+gg''=-\frac{\varphi'^2}{2}\,. \tag{A8}$$

With the results (A4) and (A7) we can now calculate the expression $A_{s_1s_2s_3s_4}\chi_{s_3s_4}$. We will also change our notation to keep the equation more compact: instead of writing flavor and spin indices explicitly, we write $\chi(1,2)$ for the quark-antiquark function, and $\mathbf{S}_j=\boldsymbol{\sigma}^j/2$ stands for the spin of the jth particle. The result is

$$A(\mathbf{k})\,\chi(1,2)=\left\{\left[\nabla^2-\frac{\varphi'(k)^2+\overline{\varphi}'^2(k)}{4}-\frac{g^2(k)+\overline{g}^2(k)}{k^2}\right]\right.$$
$$\left.+\frac{2}{k^2}\left[g^2(k)\,\mathbf{S}_1+\overline{g}^2(k)\,\mathbf{S}_2\right]\cdot\mathbf{L}-\frac{2\,g^2(k)\,\overline{g}^2(k)}{k^2}(\widehat{\mathbf{k}}\times\mathbf{S}_1)\cdot(\widehat{\mathbf{k}}\times\mathbf{S}_2)\right\}\chi(1,2)\,\phi(\mathbf{k})\,. \tag{A9}$$

The second term is the spin-orbit interaction. The operator $\mathbf{L}=i\nabla_{\mathbf{k}}\times\mathbf{k}$ is the total angular momentum. The last term can be simplified with the help of

$$\epsilon_{ijm}\epsilon_{iln}=\delta_{jl}\delta_{mn}-\delta_{jn}\delta_{lm}\,, \tag{A10}$$

which implies

$$(\widehat{\mathbf{k}}\times\mathbf{S}_1)\cdot(\widehat{\mathbf{k}}\times\mathbf{S}_2)=\tfrac{2}{3}(\mathbf{S}_1\cdot\mathbf{S}_2)-[(\widehat{\mathbf{k}}\cdot\mathbf{S}_1)(\widehat{\mathbf{k}}\cdot\mathbf{S}_2)-\tfrac{1}{3}\,\mathbf{S}_1\cdot\mathbf{S}_2]\,. \tag{A11}$$

Since the function χ is an eigenstate of total spin S and of the spin of one of the quarks S^j, the scalar term is easily evaluated:

$$2\mathbf{S}_1\cdot\mathbf{S}_2\chi(1,2)=[S(S+1)-\tfrac{3}{2}]\chi(1,2)\,. \tag{A12}$$

The quantity A can thus be written as

$$A(\mathbf{k})\chi(1,2)=\left\{\nabla^2-\frac{\varphi'(k)^2+\overline{\varphi}'(k)^2}{4}+\frac{g^2(k)\,\overline{g}^2(k)-g^2(k)-\overline{g}^2(k)}{k^2}+\frac{2}{k^2}\left[g^2(k)\,\mathbf{S}_1+\overline{g}^2(k)\,\mathbf{S}_2\right]\cdot\mathbf{L}\right.$$
$$\left.-\frac{2\,g^2(k)\,\overline{g}^2(k)}{k^2}\left[\frac{S}{3}(S+1)-[(\widehat{\mathbf{k}}\cdot\mathbf{S}_1)(\widehat{\mathbf{k}}\cdot\mathbf{S}_2)-\frac{1}{3}\,\mathbf{S}_1\cdot\mathbf{S}_2]\right]\right\}\chi(1,2)\,\phi(\mathbf{k})\,. \tag{A13}$$

This result is now introduced in (A1) to obtain the bound-state equation for mesons. First notice that the spherical symmetry of (A1) implies that the angular part of the function ϕ is given by the spherical harmonics and the wave function can be written as

$$\phi(\mathbf{k})=\sum_{L,M}\begin{pmatrix}L & S & J\\ M & M_S & M_J\end{pmatrix}\,Y_{LM}(\widehat{\mathbf{k}})\,\frac{\nu_L(k)}{k}. \tag{A14}$$

The bound-state equation becomes

$$\left\{\frac{d^2}{dk^2}+M-E(k)-\overline{E}(k)-\frac{L(L+1)}{k^2}-\frac{\varphi'^2(k)+\overline{\varphi}'^2(k)}{4}+\frac{\sin\varphi(k)\,\sin\overline{\varphi}(k)-1}{k^2}+\frac{2}{k^2}[g^2(k)\,\mathbf{S}_1+\overline{g}^2(k)\,\mathbf{S}_2]\cdot\mathbf{L}\right.$$
$$\left.-\frac{2\,g^2(k)\,\overline{g}^2(k)}{k^2}\left[\frac{S}{3}(S+1)+(\widehat{\mathbf{k}}\cdot\mathbf{S}_1)(\widehat{\mathbf{k}}\cdot\mathbf{S}_2)-\frac{1}{3}\,\mathbf{S}_1\cdot\mathbf{S}_2\right]\right\}\nu(k)=0\,. \tag{A15}$$

The factor $4K_0^3/3$ does not appear because we are now measuring all momenta, energy, and masses in units of $(4/3)^{1/3}K_0$.

Let us consider now the case of baryons. The eigenstate equation (35) becomes

$$[M-3E(p_1)]\,\chi_{s_1s_2s_5}\,\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)-2K_0^3\,\chi_{s_3s_4s_5}B_{s_1s_2s_3s_4}=0\,, \tag{A16}$$

with $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0$, and

$$B_{s_1s_2s_3s_4}\equiv\nabla_{\mathbf{q}}^2\left[\psi(\mathbf{p}_1-\mathbf{q},\mathbf{p}_2+\mathbf{q},\mathbf{p}_3)u_{s_1}^\dagger(\mathbf{p}_1)u_{s_3}(\mathbf{p}_1-\mathbf{q})u_{s_2}^\dagger(\mathbf{p}_2)u_{s_4}(\mathbf{p}_2+q)\right]_{\mathbf{q}=0}\,. \tag{A17}$$

Expanding the Laplacian gives

$$B_{s_1s_2s_3s_4} = \{\delta_{s_1s_3}\delta_{s_2s_4}\nabla^2_{\mathbf{p}_{12}} + \delta_{s_2s_4}u^\dagger_{s_1}(\mathbf{p}_1)\nabla^2 u_{s_3}(\mathbf{p}_1) + \delta_{s_1s_3}u^\dagger_{s_2}(\mathbf{p}_2)\nabla^2 u_{s_4}(\mathbf{p}_2)$$
$$+2[\delta_{s_2s_4}u^\dagger_{s_1}(\mathbf{p}_1)\boldsymbol{\nabla} u_{s_3}(\mathbf{p}_1) - \delta_{s_1s_3}u^\dagger_{s_2}(\mathbf{p}_2)\boldsymbol{\nabla} u_{s_4}(\mathbf{p}_2)]\cdot\boldsymbol{\nabla}_{\mathbf{p}_{12}}$$
$$-2u^\dagger_{s_1}(\mathbf{p}_1)\boldsymbol{\nabla} u_{s_3}(\mathbf{p}_1)\cdot u^\dagger_{s_2}(\mathbf{p}_2)\boldsymbol{\nabla} u_{s_4}(\mathbf{p}_2)\}\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\ . \tag{A18}$$

The simplification of $B_{s_1s_2s_3s_4}$ is very similar to what was done for the function $A_{s_1s_2s_3s_4}$ in the meson case and the final result is

$$B\chi(1,2,3) = \left\{ \nabla^2_{\mathbf{p}_{12}} - \frac{\varphi'^2(p_1)}{2} - \frac{2g^2(p_1)}{p_1^2} + \frac{g^2(p_1)\,g^2(p_2)}{p_1p_2}\left[\frac{2}{9}S(S+1) - \frac{1}{2}\right]\right.$$
$$-\frac{2g^2(p_1)\,g^2(p_2)}{p_1p_2}\left[(\widehat{\mathbf{p}}_1\cdot\mathbf{S}_1)(\widehat{\mathbf{p}}_2\cdot\mathbf{S}_2) - \frac{1}{3}\,(\widehat{\mathbf{p}}_1\cdot\widehat{\mathbf{p}}_2)(\mathbf{S}_1\cdot\mathbf{S}_2)\right]$$
$$\left.+\frac{4g^2(p_1)}{p_1^2}\mathbf{S}_1\cdot\mathbf{L}_1 - \frac{2g^2(p_1)}{p_1^2}(\mathbf{S}_1+\mathbf{S}_2)\cdot(i\boldsymbol{\nabla}_{p_2}\times\mathbf{p}_1)\right\}\chi(1,2,3)\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\ . \tag{A19}$$

And the bound-state equation for baryons, in units such that $(4/3)^{1/3}K_0 = 1$, is

$$\left\{3E(p_1) - M - \frac{3}{2}\nabla^2_{\mathbf{p}_{12}} + \frac{3\varphi'^2(p_1)}{4} + \frac{3g^2(p_1)}{p_1^2} + \frac{g^2(p_1)\,g^2(p_2)}{p_1p_2}\left[\frac{3}{4} - \frac{1}{3}S(S+1)\right]\right.$$
$$+\frac{3g^2(p_1)\,g^2(p_2)}{p_1p_2}\left[(\widehat{\mathbf{p}}_1\cdot\mathbf{S}_1)(\widehat{\mathbf{p}}_2\cdot\mathbf{S}_2) - \frac{1}{3}\,(\widehat{\mathbf{p}}_1\cdot\widehat{\mathbf{p}}_2)(\mathbf{S}_1\cdot\mathbf{S}_2)\right]$$
$$\left.-\frac{6g^2(p_1)}{p_1^2}\mathbf{S}_1\cdot\mathbf{L}_1 + \frac{3g^2(p_1)}{p_1^2}(\mathbf{S}_1+\mathbf{S}_2)\cdot(i\boldsymbol{\nabla}_{p_2}\times\mathbf{p}_1)\right\}\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = 0\ . \tag{A20}$$

APPENDIX B

In this appendix we solve the bound-state equation (36) for baryons, using the variational method. From (36) we have

$$M = \left\langle 3E(p_1) - \frac{3}{2}\nabla^2_{\mathbf{p}_{12}} + \frac{3}{4}\varphi'^2(p_1) + \frac{3[1-\sin\varphi(p_1)]}{p_1^2}\right\rangle_{\mathbf{P}=0}$$
$$+\left[\frac{3}{4} - \frac{1}{3}S(S+1)\right]\left\langle [1-\sin\varphi(p_1)][1-\sin\varphi(p_2)]\frac{\widehat{\mathbf{p}}_1\cdot\widehat{\mathbf{p}}_2}{p_1p_2}\right\rangle_{\mathbf{P}=0}\ , \tag{B1}$$

where the following notation has been introduced:

$$\langle f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\rangle_{\mathbf{P}=0} \equiv \int d^9p\ \delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)f(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\,|\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)|^2\ . \tag{B2}$$

For the ground state we use a Gaussian trial wave function:

$$\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = \left(\frac{\sqrt{3}\alpha^2}{\pi}\right)^{3/2} e^{-\alpha^2(p_1^2+p_2^2+p_3^2)/2}\ . \tag{B3}$$

The variational parameter α is of the order of the rms radius of the baryon, and is in units of the inverse of the potential parameter $(4/3)^{1/3}K_0$. It should be clear that the four-component nature of the single-quark wave functions is contained in the vertices Θ^j; this $\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ contributes to the "radial" part of the nucleon wave function.

For functions depending only on one of the three momenta $\mathbf{p}_i$, the integral (B2) can be written as

$$\langle f(p_1)\rangle_{\mathbf{P}=0} = \frac{4}{\sqrt{\pi}}\int_0^\infty dx\ x^2 f(\widetilde{x})e^{-x^2}\ ,$$
$$\widetilde{x} \equiv \sqrt{\frac{2}{3}}\frac{x}{\alpha}. \tag{B4}$$

With this relation we can first verify that the trial function is normalized:

$$\langle 1\rangle_{\mathbf{P}=0} = 1\ . \tag{B5}$$

The Laplacian $\nabla^2_{\mathbf{p}_{12}}$ of the trial wave function is

$$\nabla^2_{\mathbf{p}_{12}}\,\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = [\alpha^4(\mathbf{p}_1-\mathbf{p}_2)^2 - 6\alpha^2]\,\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\ . \tag{B6}$$

And the quark energy (10) for the harmonic potential we use in this paper is given by [1,5]

$$E(k) = m \sin\varphi(k) + k\cos\varphi(k) - \frac{\varphi'^2(k)}{2} - \frac{\cos^2\varphi(k)}{k^2} . \tag{B7}$$

The last two equations allow us to write the first term in (B1) as

$$\text{F.T.} = \frac{9\alpha^2}{2} + \frac{4}{\sqrt{\pi}}\int_0^\infty dx \left[\frac{\sqrt{6}}{\alpha}x^3\cos\varphi_{\overline{x}} - \frac{3x^2}{4}(\varphi'_{\overline{x}})^2 + \left(3mx^2 - \frac{9\alpha^2}{2} + \frac{9\alpha^2}{2}\sin\varphi_{\overline{x}}\right)\sin\varphi_{\overline{x}}\right] e^{-x^2} . \tag{B8}$$

This last integral cannot be calculated analytically because we do not have an analytic expression for the chiral angle φ. A useful numerical algorithm to compute this kind of integral is [15]

$$\int_0^\infty f(x)\, e^{-x^2}\, dx \approx \sum_{i=0}^{n} \omega_i\, f(x_i) , \tag{B9}$$

where $x_0 = 0$, and $(x_1, \ldots, x_n)$ are the positive zeros of the Hermite polynomial H_{2n+1}. The coefficients ω_i are certain weight factors which are tabulated in Ref. [15].

The second term in (B1), which is the hyperfine splitting, can be calculated using the general expression

$$\begin{aligned}\langle(\widehat{\mathbf{p}}_1 \cdot \widehat{\mathbf{p}}_2) f(p_1) g(p_2)\rangle_{\mathbf{P}=0} &= \frac{6\sqrt{3}}{\pi}\int_0^\infty\int_0^\infty dx\, dy\, f(\overline{x})\, g(\overline{y})\, \sinh(xy)\, e^{-(x^2+y^2)} \\ &\quad - \frac{6\sqrt{3}}{\pi}\int_0^\infty\int_0^\infty dx\, dy\, xy\, f(\overline{x})\, g(\overline{y})\, \cosh(xy)\, e^{-(x^2+y^2)}, \\ \overline{x} &\equiv \frac{x}{\sqrt{2}\alpha} .\end{aligned} \tag{B10}$$

The result we obtain for the second term in (B1) is

$$\begin{aligned}\text{S.T.} = \frac{24\alpha^2}{\sqrt{\pi}}\int_0^\infty dx \sin\varphi_{\overline{x}}\, e^{-x^2} + \frac{12\sqrt{3}\alpha^2}{\pi}\int_0^\infty\int_0^\infty dx\, dy \bigg\{ &\frac{\sinh(xy)}{xy}[1 - 2\sin\varphi_{\overline{x}} + \sin\varphi_{\overline{x}}\sin\varphi_{\overline{y}}] \\ &- \cosh(xy)\sin\varphi_{\overline{x}}\sin\varphi_{\overline{y}}\bigg\} e^{-(x^2+y^2)} .\end{aligned} \tag{B11}$$

The first integral can be calculated using (B9), while for the second integral we have

$$\int_0^\infty\int_0^\infty f(x)\, g(x)\, e^{-x^2-y^2}\, dx\, dy \approx \sum_{i=0}^{n}\sum_{j=0}^{n} \omega_i\, \omega_j\, f(x_i)\, g(x_j) . \tag{B12}$$

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