NUCLEAR PROPERTIES IN A SIMPLE QUARK MODEL

by

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Curriculum vitae

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Abstract

A simple quark model of the nucleus is developed, and a deuteron-like system is studied, which in this model is a non-relativistic, one-dimensional system of 4 quarks with only two colors, inside a finite volume. The Hamiltonian is exactly solvable and its eigenstates are of the Bethe *ansatz* type. In the ground state the quarks are completely clustered, and each cluster (nucleon) is formed by two quarks in a color-singlet state. An effective hadron theory is extracted from the quark model, where the clusters are regarded as elementary particles, with a repulsive effective interaction. The effective Hamiltonian is also of the Bethe *ansatz* type and therefore exactly solvable. The effective theory is used to construct two different approximation schemes: an *impulse approximation*, and a *cluster approximation*.

The purpose of this study is to test whether the only role of quark dynamics in the nucleus is to fix the internal structure of the nucleons. The results of the quark model are considered as "experimental" data, and the results derived using the approximations are viewed as "theoretical" predictions. Some dynamical functions related to scattering experiments are investigated. Analytical expressions are obtained for these observables, using an original technique, and some selected numerical results are presented.

It is found that the momentum distribution and correlation function for quarks of different color can be reproduced with the effective hadron theory. The observables that are more sensitive to quark effects are the elastic form factor and Coulomb sum rule. The major correction to the approximations based on the effective hadron theory arises from a quark dynamical effect different from Pauli exclusion corrections: the state of the deuteron cannot be factored into a part that depends only on cluster coordinates and another part depending on the internal structure of the clusters. Based on the results of the model, we conclude that

in a real experiment these two functions of the target (i.e. elastic form factor and Coulomb sum rule) may be the most sensitive to quark effects, other than those already present in the nucleon.

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Chapter 1

Introduction

So long as big and small are merely relative concepts, it is no help to explain the big in terms of the small.

P. A. M. DIRAC, The Principles of Quantum Mechanics.

Dirac's criticism was directed toward classical mechanics, and was one of his arguments in favor of a new theory of the *small* (quantum mechanics), departing from the laws of motion of the *big*. Quantum theory describes phenomena which would be unheard of in classical mechanics. It opened a new way of looking at nature and provided the scientific community with new and challenging problems. On the other hand, problems which were successfully solved via classical mechanics became intractable with the new mechanics and were left as belonging to the realm of the big; which is a concern of classical rather than quantum mechanics (the motion of projectiles, for example).

The continued effort among some scientists to investigate deeper into smaller levels of nature, does not derive simply from a search for higher accuracy; it is an attempt to discover new and stricking effects unlike anything known today [1]. Such discoveries usually lead also to the understanding of certain processes in the *big* domain, which were previously regarded as anomalous. Even though there is a difference between big and small, a clear borderline cannot be drawn.

[1] The search for the submicroscopic is not the only way to discover novel phenomena; however, that does not undermine its important role in contemporary physics

The level of physical description that concerns us in this dissertation is quark dynamics. When quantum mechanics was introduced, the entities that it dealt with were particles such as the electron, the photon and the atomic nucleus. A few years later it was found that the atomic nucleus contains protons and neutrons; these nucleons are bound together by the strong interaction which overcomes the electromagnetic repulsion of protons. The long range of the strong interaction was explained by the one-boson-exchange mechanism.

Nowadays, the *standard model* of particle physics is constructed from quark, leptons and other particles; their interactions are described by gauge theories. The theory of strong interactions of quarks is a non-Abelian gauge theory (QCD) with gluons as gauge bosons. It has very unique features not present in other interaction theories: color confinement, asymptotic freedom, etc.

The traditional picture of the nucleus as a system of protons and neutrons interacting through meson exchange, is very successful in explaining almost all empirical phenomena known today. This seems to indicate the hadronic nature of the nucleus, under normal conditions, which would make a detailed description in terms of quark-gluon degrees of freedom a difficult and unnecessary task. Furthermore, no QCD calculations have been made at the nuclear level of energy and distances, because the theory is highly non-perturbative in that domain. Under extreme conditions, it has been predicted that deconfinement might take place and nuclear matter would undergo a phase transition into a quark-gluon plasma [2]. In the domain of the quark-gluon plasma, hadron theories are not applicable and QCD is easier to solve.

Despite the success of traditional nuclear theory in explaining empirical phenomena under normal conditions, we still hope to find novel phenomena of quark type and maybe even traces of it in our current collection of nuclear data. The existence of such effects would have important consequences due to the very different nature of QCD. Our expectation is reinforced by some recent experimental discoveries, which some authors see as a signature of quark effects: the *EMC effect* [2] and the depletion of the longitudinal response function of nuclear targets [3].

[2] A non-technical review on the subject can be found in reference [1]

The EMC effect was discovered in 1983 by the *European Muon Collaboration* group; they measured the structure function of deep inelastic scattering from iron targets and found it to be different from that of deuterium. This result was unexpected, because in that domain the impulse approximation is believed to be valid, which implies that after correcting for nucleon excess and Fermi motion, all nuclei should have the same structure function as the deuteron. Several plausible explanations of the EMC effect have been proposed (all of different nature); in one of them, the effect is explained by assuming that the radius of nucleons is bigger when they are inside heavier nuclei [4]. The same assumption can also account for the anomaly observed in the longitudinal response function of quasielastic scattering from calcium targets: its norm appears to be smaller than expected [5]. The swelling of nucleons is viewed as a quark effect; quarks are confined to a larger *bag* in the presence of other nucleons. However, it has been claimed that other explanations of the EMC effect, which do not involve quark dynamics, have the same partial success as the swelling of nucleons does (a review is found in [6]); and even the swelling of nucleons can also be explained with meson theories.

A detailed description of all nuclear phenomena, in terms of quark-gluon degrees of freedom, being almost impossible and of doubtful relevance, we resort to the use of simple models to look for phenomena which cannot be explained in terms of traditional nuclear physics. A few models have been used to look for quark effects in the nucleus; we will call them hereafter *cluster models.* There is another model which provides a more complete dynamical description of the nucleus in terms of quarks, known as the *flip-flop* model[7]. It constitutes a true *quark model* in contrast with cluster models. The subject of this dissertation is the study of a quark model different from the *flip-flop* model.

In cluster models [8,9,10], the nucleus is viewed as a system of nucleons; each nucleon is a cluster of quarks. The traditional picture to be tested is described by a nuclear wave function which is a product of nucleonic and cluster functions; for example, if there were only two nucleons and each one had only two valence quarks, the wave function would be

$$\psi(x_1,x_2,x_3,x_4) = \phi\left(\frac{x_1+x_2}{2},\frac{x_3+x_4}{2}\right)\,\chi(x_1,x_2)\,\chi(x_3,x_4), \tag{1.1}$$

where (x_1,x_2,x_3,x_4) are the positions of the four quarks, $\phi(x,y)$ is the wave function of two nucleons, and $\chi(x_1,x_2) = \chi(x_1-x_2)$ the wave function of the two quarks inside a cluster.

Even though ψ is a quark wave function, it does not lead to any new quark effects, other than those already present in the nucleon; the internal quark structure of each cluster remains always the same. The only role of quark dynamics in traditional nuclear physics is to fix the internal properties of the nucleons which have been incorporated through the function χ. This picture has many implicit assumptions which may fail; the main source of its failure is expected to be that the wave function ψ is not completely antisymmetric under the exchange of any two quarks, as required by the exclusion principle (even if ϕ and χ have the correct symmetries, ψ does not). Quark antisymmetrization of the nuclear wave function introduces exchange terms which make the internal structure of clusters no longer invariant. Cluster models are developed by assuming some wave function of the type of equation (1.1), which is then antisymmetrized under exchange of quarks.

A more detailed description of quark dynamics inside the nucleus is provided by a quark model[3]. A quark Hamiltonian is constructed, which preserves some general features of QCD. The quark Hamiltonian is then solved, and its eigenstates are used to calculate dynamic properties of the nucleus. A hadron theory is also extracted, by deriving an effective nucleon-nucleon potential from the quark model. The eigenstates of the effective hadron Hamiltonian can also be used to construct quark wave functions such as equation (1.1). The advantage over cluster models is that these functions are not simply assumed; they are actually calculated in the effective theory. The assumption that the nucleus is a system of independent nucleons can be tested by comparing with the results from the quark Hamiltonian, and effects of quark antisymmetrization are also investigated as in clusters models.

The *flip-flop* model Hamiltonian confines the nearest-neighbor quarks in each cluster separately, when the clusters are in color-singlet states (when they are not, six quarks can also cluster together). With an adiabatic theory, the effective cluster-cluster potential is extracted. The effective potential is used to construct a hadronic Hamiltonian whose eigenfunctions ϕ lead to cluster approximations of the kind of equation (1.1). Those approximations have been tested both in small nuclei and in nuclear matter [11,12,13,14,15,16].

[3]The term *quark-cluster model* is frequently found in the literature. To avoid confusion with the first kind of models, we call them simply quark models

The quark model that we introduce in this dissertation is an extension into finite nuclei of the model for nuclear matter proposed by Koltun *et al* [17,18,19,20]. In this model, the one-dimensional Hamiltonian is exactly solvable and its solutions are of the Bethe *ansatz* type (chapter 4). An effective hadron theory which reproduces the energy spectrum very closely, is extracted from the model; in this effective theory the nucleus is a system of nucleons. Two different approximation schemes will be constructed using the effective theory. In the first approximation, the internal structure of nucleons is considered to be completely independent of the presence of other nucleons; wave functions of the form of equation (1.1) are then used. We will refer to this approximation as the impulse approximation. The second approximation —cluster approximation— also assumes a nucleus formed by separate nucleons, but their internal structure is affected by other nucleons through quark exchange. The states of the system are of quark nature, derived from a combination of nucleon effective wave functions and quark states of the effective nucleon.

The model does not constitute a realistic theory of nuclear dynamics in terms of quark degrees of freedom. Yet it is a useful tool in the search for quark effects in the nucleus, other than those already present at the nucleon level; this model retains the minimun aspects of quark dynamics that —in our opinion– may lead to quark effects. For the purpose of our study, the quark model is regarded as if it was an accurate representation of the real world; its results are considered as "experimental" data. The results derived from the approximations based on the effective theory are given the status of "theoretical" predictions. If the only role of quark dynamics in the nucleus is to fix the internal structure of the nucleons, then the "theoretical" predictions should agree with the "experimental" data. This hypothesis will be tested by investigating the behavior of some dynamical functions related to scattering experiments, for a deuteron-like target.

The kind of questions that we will investigate are mostly the same that have been addressed in the *flip-flop* model. However, we use different methods; our system allows us to compute observables analytically, while in the *flip-flop* model Monte Carlo techniques have been used. The results we obtain also show some differences with the *flip-flop* model results as discussed in chapter 7.

A mathematical definition of the observables that we will study is found in chapter 2, and their relation to scattering cross sections is discussed in chapter 3. The observables of interest are:

1. **Quark momentum distribution**: the probability of finding a quark with some given momentum, inside the nucleus. It can be measured in deep-inelastic scattering experiments.

2. **Elastic form factor**: Fourier transform of the density of quarks in the nucleus, which is related to the elastic scattering cross section.

3. **Quark correlation function**: Fourier transform of the probability of finding two quarks at a given distance from each other. In order to discuss the results in a more convenient way, it will be separated into two parts: same-color and different-color. The quark correlation function can be measured indirectly in inelastic scattering experiments; its separate terms could only be measured with probes sensitive to color, which is not the case in electron scattering.

4. **Coulomb sum rule**: The sum of the response function (chapter 3) over all possible energy transfers. It is a combination of the correlation function and the elastic form factor, which can be measured in inelastic scattering processes.

Bethe *ansatz* models are usually formulated in coordinate space; eigenstates and eigenvalues are derived in an elegant way, but this formulation makes the calculation of correlations an intractable problem [21]. Kebukawa [22] proposes a different approach by introducing second quantization in momentum space. Using Kebukawa's wave function, the observables that we are interested in can be calculated analytically as shown in chapter 6.

The momentum representation of the Bethe *ansatz* given in reference [22] is only valid for fermions with two colors, in which case only clusters of two quarks are obtained. In the quark model that we use, it has been found [20] that the approximate ground state obtained from a variational approach has very similar features in the cases of two and three colors; in spite of the fact that clusters of two quarks are quasibosons, while clusters of three quarks are

quasifermions. The reason for that is the effective delta-function repulsion among clusters, which makes the spectrum look the same whether the particles are fermions or bosons. In view of this, we have decided to restrict our treatment to the two color case; in the future it may be interesting to extend Kebukawa's states to the three-color case and to verify whether the results are in fact very similar.

Our work is contained mainly in chapters 5 through 7, and in the appendices. Chapters 2 and 3 are a review of some results, most of which are standard. The main reason for including such a survey is to define a consistent notation that will be used in later chapters. In chapter 4 we review the Bethe *ansatz* technique as applied to a system of particles with delta-function interaction; by the end of the chapter (and in appendix C), we give a simplified form of the Bethe *ansatz* for two clusters, which we have derived from the results of Kebukawa[22].

The quark model that we use, and the effective hadronic theory are introduced in chapter 5. Chapter 6 and appendices A and B give the detailed calculation of observables. In chapter 7 we show the results obtained; their implications are discussed in chapter 8, where conclusions and future extensions are outlined.

Chapter 2

Basic tools

In this chapter we summarize some results from many-body theory. A detailed explanation of most of the topics covered here has been given in several textbooks in many-body theory; for example references [23] and [24]. The main purpose of this short summary is to define a consistent notation that will be used throughout the next chapters.

2.1 The symmetric group

Permutations of the symmetric group S_n will be written in the form

$$\nu = (\nu_1, \nu_2, \ldots, \nu_n), \tag{2.1}$$

where the n coordinates ν_i are all integers from 1 to n, none of them repeated. The product of two permutations, $\mu\nu$, is another permutation, σ, whose coordinates are

$$\sigma = \mu\nu \equiv (\mu_{\nu_1}, \mu_{\nu_2}, \ldots, \mu_{\nu_n}). \tag{2.2}$$

A representation of the symmetric group S_n over the space of n-variable functions, is defined in the following way

$$\widehat{\nu}\ f(x_1, \ldots, x_n) \equiv f(x_{\nu_1}, \ldots, x_{\nu_n}); \tag{2.3}$$

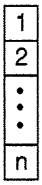
and if the functions carry also some internal indices α_i, a representation of S_n is

$$\widehat{\nu}\ f_{\alpha_1,\ldots,\alpha_n}(x_1, \ldots, x_n) \equiv f_{\alpha_{\nu_1},\ldots,\alpha_{\nu_n}}(x_{\nu_1}, \ldots, x_{\nu_n}). \tag{2.4}$$

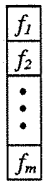
The *normal antisymmetrizer* [25, chapter 5] $A_n(1,\ldots,n)$ is defined by

$$A_n(1,2,\ldots,n) \equiv \frac{1}{n!}\sum_{\nu\in S_n}(-1)^\nu \widehat{\nu}, \tag{2.5}$$

where $(-1)^\nu$ stands for the parity of ν. $A_n(1,\ldots,n)$ is the antisymmetrizer corresponding to the *standard tableau*



for a *normal tableau* of the form



$$0 < f_1 < f_2 < \cdots < f_m \leq n,$$

the antisymmetrizer is defined as

$$A_n(f_1,f_2,\ldots,f_m) \equiv \frac{1}{m!}\widetilde{\sum_{\nu\in S_n}}(-1)^\nu \widehat{\nu}; \tag{2.6}$$

the tilde above the sum indicates that it runs only over permutations of the indices which are in the *tableau*; namely, the $m!$ permutations of S_n for which $\nu_j = j$, if j is not equal to any of the indices f_i of the *tableau*. For *tableaux* with more than one column the antisymmetrizers are obtained as the product of the antisymmetrizers of each column.

The permutations of m numbers $f_1,\ldots,f_m$ can be generated from the permutations of the $m-1$ first numbers by transposing the last one with each of the elements of each permutation; this implies the following recurrence relation for antisymmetrizers

$$A_n(f_1,\ldots,f_m) = \frac{1}{m}\left[\mathbf{1}_n - \sum_{j=1}^{m-1}\widehat{T}_{f_j,f_m}\right] A_n(f_1,\ldots,f_{m-1}), \tag{2.7}$$

where $\widehat{T}_{j,l}$ is the representation of a simple transposition of the indices j and l, and $\mathbf{1}_n$ is the

identity. A direct consequence of this is

$$\begin{aligned} A_n(f_1,\ldots,f_m)A_n(f_1,\ldots,f_{m-1}) &= \frac{1}{m}\left[\mathbf{1}_n - \sum_{j=1}^{m-1}\widehat{T}_{f_j,f_m}\right] A_n(f_1,\ldots,f_{m-1}) \\ &= A_n(f_1,\ldots,f_m). \end{aligned} \tag{2.8}$$

A symmetrizer is also defined in a similar way as the antisymmetrizer; for example, for a normal *tableau*, $(f_1, f_2, \ldots, f_m)$, the symmetrizer is

$$S_n(f_1, f_2, \ldots, f_m) \equiv \frac{1}{m!}\widetilde{\sum_{\nu\in S_n}}\widehat{\nu}. \tag{2.9}$$

2.2 Second quantization

In second quantization, the states of a system of particles are vectors in a collection of Hilbert spaces for zero-particles, one-particle, two-particles, etc. The complete space is spanned by a set of field operators acting on the vacuum state; and the conditions imposed by the exclusion principle are built in the algebra of the field operators.

2.2.1 Field operators

Consider a field with some internal degree of freedom, in one dimensional space. The field is described by an operator $\phi_\alpha(x)$ and its hermitian conjugate $\phi^\dagger_\alpha(x)$; if the field is fermionic, the following anticommutation relations must be imposed

$$\begin{cases} \{\phi_\alpha(x), \phi^\dagger_\beta(y)\} = \delta_{\alpha,\beta}\,\delta(x-y), \\ \{\phi_\alpha(x), \phi_\beta(y)\} = \{\phi^\dagger_\alpha(x), \phi^\dagger_\beta(y)\} = 0. \end{cases} \tag{2.10}$$

The operator $\phi^\dagger_\alpha(x)$ *creates* a particle with internal quantum numbers α [1], at the position x; and $\phi_\alpha(x)$ *annihilates* the same particle.

If $|\,0\rangle$ is the vacuum state, a basis for the subspace of n particles is given by the vectors of the form

$$|\,F\rangle = \int_{-\infty}^{+\infty} d^n x \sum_{\alpha_1,\ldots,\alpha_n} F_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n)\,\phi^\dagger_{\alpha_1}(x_1)\cdots\phi^\dagger_{\alpha_n}(x_n)\,|\,0\rangle, \tag{2.11}$$

[1] α can represent several internal quantum numbers such as spin, isospin, flavor, color, etc.

where $F_{\alpha_1,\ldots,\alpha_n}$ is an n-variable function which we will call the *wave function* [2] of the state $|F\rangle$. This wave function is not required to have any symmetry at all, because the antisymmetry under exchange of two particles is already implied by the operator algebra.

Since the n coordinates $(x_1,\ldots,x_n)$ and the n quantum numbers $(\alpha_1,\ldots,\alpha_n)$ are all dummy indices, the vector $|F\rangle$ can be written as

$$|F\rangle = \int_{-\infty}^{+\infty} d^n x \sum_{\alpha_1,\ldots,\alpha_n} F_{\alpha_{\nu_1},\ldots,\alpha_{\nu_n}}(x_{\nu_1},\ldots,x_{\nu_n})\, \phi^\dagger_{\alpha_{\nu_1}}(x_{\nu_1})\cdots\phi^\dagger_{\alpha_{\nu_n}}(x_{\nu_n})\,|0\rangle, \tag{2.12}$$

where ν is any permutation of S_n. Using the anticommutation property for creation operators we have

$$|F\rangle = (-1)^\nu \int_{-\infty}^{+\infty} d^n x \sum_{\alpha_1,\ldots,\alpha_n} \widehat{\nu}\,[F_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n)]\; \phi^\dagger_{\alpha_1}(x_1)\cdots\phi^\dagger_{\alpha_n}(x_n)\,|0\rangle; \tag{2.13}$$

since the wave function does not necessarily have any symmetry, in principle there are $n!$ wave functions $\widehat{\nu}F$ which correspond to the same state $|F\rangle$.

Another consequence of equation (2.13) is that any nontrivial wave function which is symmetric under the exchange of two of its arguments leads to a vector $|F\rangle$ that vanishes. Namely, if the wave function has the property:

$$A_n(1,\ldots,n)\;[F_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n)] = 0, \tag{2.14}$$

then the vector $|F\rangle$ is zero.

2.2.2 Momentum representation

We will now find the momentum representation of the n-particle vectors. Only the discrete case will be discussed, which is obtained if the space is finite

$$-\frac{L}{2} \le x \le \frac{L}{2}. \tag{2.15}$$

Later, when the Hamiltonian of the system is solved, it will be necessary to introduce periodic boundary conditions. It will then prove useful to use momentum representation; to rewrite

[2] We are using the notation of reference [26], in the non-relativistic case

the states in momentum representation, we first write the field operators $\phi_\alpha(x)$ and $\phi^\dagger_\alpha(x)$ as Fourier series

$$\phi_\alpha(x) = \frac{1}{\sqrt{L}} \sum_p \exp\left(i\frac{2\pi}{L}px\right) A_{\alpha,p}, \tag{2.16}$$

$$\phi^\dagger_\alpha(x) = \frac{1}{\sqrt{L}} \sum_p \exp\left(-i\frac{2\pi}{L}px\right) A^\dagger_{\alpha,p}, \tag{2.17}$$

where the sums run over all integers p and the momentum operators $A_{\alpha,p}$ and $A^\dagger_{\alpha,p}$ have been defined as

$$A_{\alpha,p} = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx \, \exp\left(-i\frac{2\pi}{L}px\right) \phi_\alpha(x), \tag{2.18}$$

$$A^\dagger_{\alpha,p} = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx \, \exp\left(i\frac{2\pi}{L}px\right) \phi^\dagger_\alpha(x). \tag{2.19}$$

The choice of a factor $1/\sqrt{L}$ in the Fourier series was made in order to obtain dimensionless operators $A^\dagger$ and A. The operator $A^\dagger_{\alpha,p}$ creates a particle with momentum $2\pi p/L$ and internal quantum numbers α; $A_{\alpha,p}$ annihilates the same particle. Their anticommutators are derived from equations (2.10)

$$\begin{cases} \{A_{\alpha,p}, A^\dagger_{\beta,q}\} = \delta_{p,q}\delta_{\alpha,\beta}, \\ \{A_{\alpha,p}, A_{\beta,q}\} = \{A^\dagger_{\alpha,p}, A^\dagger_{\beta,q}\} = 0. \end{cases} \tag{2.20}$$

If the field operators in the n-particle state, equation (2.11), are replaced by their Fourier series, the state becomes

$$|F\rangle = \sum_{p_1,\ldots,p_n} \sum_{\alpha_1,\ldots,\alpha_n} F_{\alpha_1,\ldots,\alpha_n}(p_1,\ldots,p_n) A^\dagger_{\alpha_1,p_1} \cdots A^\dagger_{\alpha_n,p_n} |0\rangle, \tag{2.21}$$

where $F_{\alpha_1,\ldots,\alpha_n}(p_1,\ldots,p_n)$ are the Fourier coefficients of the wave function

$$F_{\alpha_1,\ldots,\alpha_n}(p_1,\ldots,p_n) \equiv \frac{1}{L^{n/2}} \int_{-L/2}^{L/2} d^n x \, \exp -i\frac{2\pi}{L}(p_1x_1 + \cdots + p_nx_n) \, F_{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n). \tag{2.22}$$

2.2.3 Density and correlation operators

In this section we will assume periodic boundary conditions; therefore, the domain of definition of the field operators can be extended beyond the interval $[-L/2, L/2]$, according to the

periodicity condition

$$\phi_\alpha(x) = \phi_\alpha(x + mL), \tag{2.23}$$

for any integer m.

We will define here density and correlation operators, which will be used in a later section when matrix elements of one and two-body operators are calculated. The one-particle density operator is defined as

$$\widehat{\rho}(x) \equiv \sum_\alpha \phi^\dagger_\alpha(x)\, \phi_\alpha(x); \tag{2.24}$$

if the system is in an n-particle state $\mid F\rangle$, the probability of finding one particle at the point x is given by [3]

$$\rho(x) \equiv \frac{1}{n}\langle F \mid \widehat{\rho}(x) \mid F\rangle. \tag{2.25}$$

The Fourier series expansion for the one-particle density can be written in the form

$$\widehat{\rho}(x) = \frac{1}{L}\sum_q \exp\left(-i\frac{2\pi}{L}qx\right)\ \widehat{\rho}(q), \tag{2.26}$$

where the momentum representation of the density operator is

$$\begin{aligned}\widehat{\rho}(q) &= \int_{-L/2}^{L/2} dx\ \exp\left(i\frac{2\pi}{L}qx\right)\ \widehat{\rho}(x) \\ &= \sum_{\alpha,p} A^\dagger_{\alpha,p+q} A_{\alpha,p}.\end{aligned} \tag{2.27}$$

When the operator $\widehat{\rho}(q)$ acts on a state of total momentum Q, the result is a state of total momentum $Q + q$; therefore, its matrix elements are non-zero only if the total momentum of the final state equals the initial momentum plus q. The *elastic form factor* is defined as

$$F(q) \equiv \frac{1}{n}\langle i_q \mid \widehat{\rho}(q) \mid i\rangle, \tag{2.28}$$

where the state $\mid i_q\rangle$ has the same internal structure as $\mid i\rangle$, but their total momenta differ by q. If $\mid i\rangle$ and $\mid f\rangle$ are both states of n particles, with total momenta differing by q, but $\mid f\rangle \neq \mid i_q\rangle$, the *inelastic form factor* is defined by

$$F_{f,i}(q) \equiv \frac{1}{n}\langle f \mid \widehat{\rho}(q) \mid i\rangle. \tag{2.29}$$

[3] The states are assumed to be normalized; if they are not, a factor of $1/\langle F \mid F\rangle$ must be included.

Another useful operator is the correlation operator defined as

$$\widehat{P}(x) \equiv \int_{-L/2}^{L/2} dy \sum_{\alpha,\beta} \phi_\alpha^\dagger(y)\, \phi_\beta^\dagger(y+x)\, \phi_\beta(y+x)\, \phi_\alpha(y). \tag{2.30}$$

From equation (2.23) it follows that $\widehat{P}(x)$ has also period L; furthermore,

$$\widehat{P}(x) = \widehat{P}(L-x); \tag{2.31}$$

therefore, it is enough to consider only the interval $0 \leq x \leq L/2$.

In the n-particle state $|\, F\rangle$, the probability of finding two particles at a distance x from each other is

$$P(x) = \frac{1}{n(n-1)} \langle F|\, \widehat{P}(x)\, |F\rangle. \tag{2.32}$$

The Fourier series for the operator $\widehat{P}(x)$ is

$$\widehat{P}(x) = \frac{1}{L} \sum_q \exp\left(-i\frac{2\pi}{L} qx\right)\, \widehat{P}(q), \tag{2.33}$$

where its momentum representation is given by

$$\begin{aligned} \widehat{P}(q) &= \int_0^L dx\, \exp\left(i\frac{2\pi}{L} qx\right)\, \widehat{P}(x) \\ &= \sum_{\alpha,\beta,p,r} A^\dagger_{\alpha,p-q} A^\dagger_{\alpha,r+q} A_{\beta,r} A_{\alpha,p}. \end{aligned} \tag{2.34}$$

Momentum density operators can also be defined. Of particular interest to us is the one-particle momentum density operator

$$\widehat{N}(p) \equiv \sum_\alpha A^\dagger_{\alpha,p} A_{\alpha,p}, \tag{2.35}$$

whose expectation value leads to the momentum distribution

$$N(p) = \frac{1}{n} \langle F|\, \widehat{N}(p)\, |F\rangle. \tag{2.36}$$

The momentum distribution $N(p)$ equals the probability of finding a particle moving with momentum $2\pi p/L$.

2.3 Symmetries

2.3.1 Galilean invariance

If the system has translational invariance, then its states are eigenstates of the total momentum operator. This implies that the wave function $F_{\alpha_1,\ldots,\alpha_n}(p_1,\ldots,p_n)$ becomes zero unless the sum of the momenta $p_1,\ldots,p_n$ equals a constant Q (the total momentum of the system).

$$F_{\alpha_1,\ldots,\alpha_n}(p_1,\ldots,p_n) = \delta(p_1+\cdots+p_n-Q)\; F_{\alpha_1,\ldots,\alpha_n}(p_1,\ldots,p_n). \tag{2.37}$$

Galilean invariance states that if the momenta of all of the particles in the system were increased by the same quantity, it would remain the same except for an overall increase in the total momentum. In the language of second quantization this means that the state has the form

$$|F\rangle = \sum_{\alpha_1,\ldots,\alpha_n} \sum_{p_1,\ldots,p_n} \delta(p_1+\cdots+p_n-Q)\; F_{\alpha_1,\ldots,\alpha_n}(p_1-Q/n,\ldots,p_n-Q/n)\; A^\dagger_{\alpha_1,p_1}\cdots A^\dagger_{\alpha_n,p_n}\,|0\rangle, \tag{2.38}$$

where Q is the total momentum; here we have assumed periodic boundary conditions. In the *center of momentum* frame —the frame where the total momentum is zero— the states can be written as

$$|F\rangle = \sum_{\alpha_1,\ldots,\alpha_n} \sum_{p_1,\ldots,p_n} \delta(p_1+\cdots+p_n)\; F_{\alpha_1,\ldots,\alpha_n}(p_1,\ldots,p_n)\; A^\dagger_{\alpha_1,p_1}\cdots A^\dagger_{\alpha_n,p_n}\,|0\rangle, \tag{2.39}$$

here the sums over p_i no longer run over the integers, but over all rational numbers which are equal to an integer plus $(Q \bmod n)/n$. Likewise the second indices of the field operators are not integers anymore but rational.

2.3.2 Color symmetry

We will now restrict our treatment to particles with only one internal degree of freedom which we will call color; this quantum number takes on a finite number of discrete values $\alpha_1,\alpha_2,\ldots,\alpha_j$. Consider the two-color case ($j=2$); we will introduce the simplified notation

$$a^\dagger_p \equiv A^\dagger_{\alpha_1,p} \qquad\qquad b^\dagger_p \equiv A^\dagger_{\alpha_2,p}; \tag{2.40}$$

and similarly for annihilation operators. Let us define the color operators

$$C_+^a \equiv \sum_p a_p^\dagger b_p, \tag{2.41}$$

$$C_-^a \equiv \sum_p b_p^\dagger a_p, \tag{2.42}$$

$$C_3^a \equiv \frac{1}{2}\sum_p \left(a_p^\dagger a_p - b_p^\dagger b_p\right); \tag{2.43}$$

these operators have the following commutation relations

$$[C_+^a, C_-^a] = 2C_3^a, \tag{2.44}$$

$$[C_3^a, C_\pm^a] = \pm C_\pm^a, \tag{2.45}$$

which correspond to the standard form of the $SU(2)$ Lie algebra [27, chapter 6]. Since the rank of this algebra is 1, there is one Casimir operator which commutes with the Cartan subalgebra (in this case the operator C_3^a); the Casimir operator has the form

$$C^2 \equiv \frac{1}{2}\left[C_+^a C_-^a + C_-^a C_+^a\right] + C_3^a C_3^a. \tag{2.46}$$

A system has color invariance if its Hamiltonian commutes with both C^2 and C_3^a. In that case there exists a set of eigenstates of energy which are also eigenstates of the color operators C^2 and C_3^a. We will now investigate what conditions must be satisfied by states of the form of equation (2.21), in order to be eigenstates of C^2 and C_3^a. In first quantization, the general conditions that the wave function must satisfy are well known [4]; we will find their equivalent in second quantization (see theorem below).

In the simplified notation, the operators $A_{\alpha,p}^\dagger$ in equation (2.21) can be either $a_p^\dagger$ or $b_p^\dagger$. All of the terms of the sum which have the same number of operators $a^\dagger$ and $b^\dagger$ can be grouped together into one term; for instance, for $n = 3$ the three terms which have only one operator $a^\dagger$ can be added together in the following way

$$\sum_{p_1,p_2,p_3}\left[F_{abb}(p_1,p_2,p_3)a_{p_1}^\dagger b_{p_2}^\dagger b_{p_3}^\dagger + F_{bab}(p_1,p_2,p_3)b_{p_1}^\dagger a_{p_2}^\dagger b_{p_3}^\dagger + F_{bba}(p_1,p_2,p_3)b_{p_1}^\dagger b_{p_2}^\dagger a_{p_3}^\dagger\right]$$
$$= \sum_{p_1,p_2,p_3} f_1(p_1,p_2,p_3)\, a_{p_1}^\dagger b_{p_2}^\dagger b_{p_3}^\dagger,$$

[4] See for example the paper by Lieb & Mattis[28].

where the function $f_1(p_1,p_2,p_3)$ has been defined as

$$f_1(p_1,p_2,p_3) \equiv F_{abb}(p_1,p_2,p_3) - F_{bab}(p_2,p_1,p_3) - F_{bba}(p_3,p_2,p_1). \tag{2.47}$$

The complete state, equation (2.21), contains also terms with two or three operators a, or with none at all; therefore, grouping terms together as was done in equation (2.47), we can write the state in the form

$$\begin{aligned} |F\rangle = \sum_{p_1,p_2,p_3} \Big[& f_0(p_1,p_2,p_3)\, b^\dagger_{p_1} b^\dagger_{p_2} b^\dagger_{p_3} \\ & + f_1(p_1,p_2,p_3)\, a^\dagger_{p_1} b^\dagger_{p_2} b^\dagger_{p_3} + f_2(p_1,p_2,p_3)\, a^\dagger_{p_1} a^\dagger_{p_2} b^\dagger_{p_3} + f_3(p_1,p_2,p_3)\, a^\dagger_{p_1} a^\dagger_{p_2} a^\dagger_{p_3} \Big] |0\rangle. \end{aligned} \tag{2.48}$$

In general, an n-particle state can then be written in the form

$$|F\rangle = \sum_{p_1,\ldots,p_n} \sum_{m=0}^{n} f_m(p_1,\ldots,p_n)\, a^\dagger_{p_1} \cdots a^\dagger_{p_m} b^\dagger_{p_{m+1}} \cdots b^\dagger_{p_n} |0\rangle; \tag{2.49}$$

each term in the sum over m is an eigenstate of the operator C_3^a with eigenvalue [5] $m - n/2$:

$$|\psi_m\rangle \equiv \sum_{p_1,\ldots,p_n} \psi(p_1,\ldots,p_n) a^\dagger_{p_1} \cdots a^\dagger_{p_m} b^\dagger_{p_{m+1}} \cdots b^\dagger_{p_n} |0\rangle, \tag{2.50}$$

$$C_3^a |\psi_m\rangle = \left(m - \frac{n}{2}\right) |\psi_m\rangle. \tag{2.51}$$

In a similar way as in equation (2.13) these states can also be written as

$$|\psi_m\rangle = (-1)^\nu \sum_{p_1,\ldots,p_n} [\hat{\nu}\, \psi(p_1,\ldots,p_n)]\, a^\dagger_{p_1} \cdots a^\dagger_{p_m} b^\dagger_{p_{m+1}} \cdots b^\dagger_{p_n} |0\rangle, \tag{2.52}$$

where ν is any permutation that does not mix the first m indices with the other $n-m$ ($\nu_i \leq m$ for $i \leq m$). The state $|\psi_m\rangle$ vanishes if and only if

$$\chi_m(p_1,\ldots,p_n) = 0, \tag{2.53}$$

for any value of the arguments p_i, where the function χ_m is given by

$$\chi_m(p_1,\ldots,p_n) \equiv A_n(1,\ldots,m)\, A_n(m+1,\ldots,n)\, \psi(p_1,\ldots,p_n). \tag{2.54}$$

The state $|\psi_m\rangle$ will also be an eigenstate of the Casimir operator C^2 if the wave function has certain symmetries given by the following theorem.

[5] This follows from the definition of C_3^a which is simply one half of the number of particles with color a minus the number of particles with color b.

Theorem 1 *Let*

$$|\psi\rangle = \sum_{p_1} \cdots \sum_{p_n} \psi(p_1, \ldots, p_n) a^\dagger_{p_1} \cdots a^\dagger_{p_m} b^\dagger_{p_{m+1}} \cdots b^\dagger_{p_n} |0\rangle, \tag{2.55}$$

be a non-zero state. Then $|\psi\rangle$ *is an eigenstate of* C^2

$$C^2 |\psi\rangle = \left(l - \frac{n}{2}\right)\left(l - \frac{n}{2} + 1\right) |\psi\rangle, \tag{2.56}$$

if, and only if the following two conditions are true

$$A_n(1, \ldots, l)\, A_n(m+1, \ldots, n)\, \psi(p_1, \ldots, p_n) \neq 0, \tag{2.57}$$

$$A_n(1, \ldots, l+1)\, A_n(m+1, \ldots, n)\, \psi(p_1, \ldots, p_n) = 0, \tag{2.58}$$

for some integer l *in the interval* $n/2 \leq l \leq n$ *(if* $l = n$ *the second condition should be ignored).*

The proof will be given at the end of this section. Since the state is not zero, from equation (2.53) it follows that the second condition in the theorem can only be satisfied if $l \geq m$. If the state $|\psi\rangle$ is an eigenstate of C^2 with eigenvalue as given by the theorem, it is said to have color $l - n/2$; the states of color $l - n/2$ form an irreducible color representation of dimension $(2l - n + 1)$. For a given integer m, the state $|\psi_m\rangle$ can belong to irreducible color representations of color $|m - n/2|$, $|m - n/2| + 1$, ..., $n/2$, depending on the symmetries of the wave function. If the number of particles n is even, only integer color representations are attained, and if there is an odd number of particles, the representations obtained have all half-integer color.

The irreducible representation to which the state $|\psi_m\rangle$ belongs can be represented by a *tableau* with n blocks. The *tableaux* that represent the irreducible representations of the group $SU(2)$ have only two rows, and the dimension of the corresponding representation is $(\lambda_1 - \lambda_2 + 1)$, where λ_1 and λ_2 are the number of blocks of the first and second row [6]. Since $|\psi_m\rangle$ belongs to a $(2l - n + 1)$ dimensional representation, we must construct the corresponding *tableau* with l blocks in the first row and the other $(n - l)$ in the second. By

[6] See for example reference [29], chapter 13.

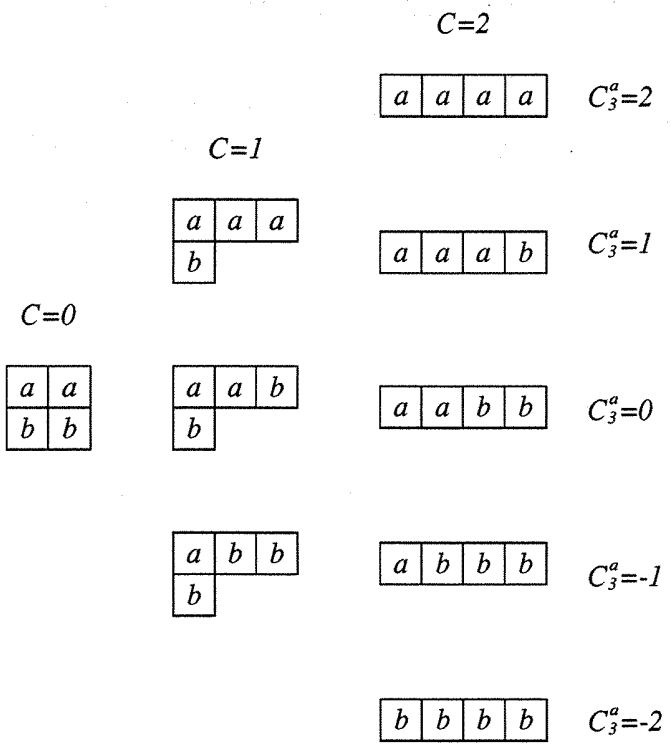


Figure 2-1: SU(2) irreducible representations for 4–particle states.

convention, the first m blocks (from left to right and downwards) are assigned an index a and the rest an index b.

In figure 2.1 we show the case of four particles. The three columns in figure 2.1 are a color singlet, triplet and quintet. Each row corresponds to an eigenvalue of C_3^a; the *tableaux* for negative values of C_3^a are derived from those with positive C_3^a by exchanging a's and b's. All of the allowed *tableaux* can be obtained following the rules:

1. On each column the color indices cannot be repeated, and they should appear in order: first a and then b. This follows from the condition $l \geq m$.

2. The color indices on each row can be repeated but the a's must go before the b's. This is a consequence of our convention to move all operators $a^\dagger$ to the left of the operators $b^\dagger$.

3. The eigenvalues corresponding to a given *tableau* are:

$$C_3^a = \frac{1}{2}(\#a's - \#b's), \tag{2.59}$$

$$C = \frac{1}{2}(\#\text{blocks on row 1} - \#\text{blocks on row 2}). \tag{2.60}$$

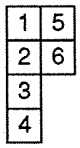
The theorem above can be explained in simple terms in the language of *tableaux*. For example consider the states of six particles, three of each color

$$|\psi_3\rangle = \sum_{p_1} \cdots \sum_{p_6} \psi(p_1,p_2,p_3,p_4,p_5,p_6) a^\dagger_{p_1} a^\dagger_{p_2} a^\dagger_{p_3} b^\dagger_{p_4} b^\dagger_{p_5} b^\dagger_{p_6} |0\rangle; \tag{2.61}$$

following the rules above, 4 different tableaux can be formed with 3 labels a and three b's; one of them is  , which belongs to a color triplet. To obtain the member of the triplet with $C_3^a = 0$, we must use wave functions ψ such that the function

$$\chi(p_1,\ldots,p_6) \equiv A_6(1,2,3,4)\, A_6(4,5,6)\psi(p_1,\ldots,p_6),$$

has the symmetries given by the normal *tableau*



namely, $\chi(p_1,p_2,p_3,p_4,p_5,p_6)$ is completely antisymmetric under the exchange of the indices (p_1,p_2,p_3,p_4) and on the indices (p_5,p_6), but it cannot be antisymmetrized under the indices (p_1,p_2,p_3,p_4,p_5). Notice that the *tableau* that specifies the symmetries of the wave function was obtained by reflecting the *tableau* that defines the color representation through the plane $y = -x$, and replacing the color indices by the numbers from one to six. We will now proof the theorem.

Proof of theorem 1: We have already mentioned that the state of equation (2.50) is an eigenstate of C_3^a, with eigenvalue $m - n/2$; therefore, in order to prove that it is also an eigenstate of C^2 with eigenvalue $C = l - n/2$ it is sufficient to prove that

$$\begin{cases} (C_+^a)^{l-m} |\psi\rangle \neq 0, \\ (C_+^a)^{l-m+1} |\psi\rangle = 0, \end{cases} \tag{2.62}$$

Using equation (2.52), the state can be written as

$$|\psi\rangle = \sum_{p_1} \cdots \sum_{p_n} \chi(p_1, \ldots, p_n) a^\dagger_{p_1} \cdots a^\dagger_{p_m} b^\dagger_{p_{m+1}} \cdots b^\dagger_{p_n} |0\rangle, \tag{2.63}$$

where the function χ is defined by

$$\chi(p_1, \ldots, p_n) \equiv A_n(1, \ldots, m)\, A_n(m+1, \ldots, n)\, \psi(p_1, \ldots, p_n). \tag{2.64}$$

Using the definition of the operator C^a_+ we have

$$\begin{aligned} C^a_+ |\psi\rangle &= \sum_{j=m+1}^{n} \sum_{p_1,\ldots,p_n} \chi(p_1, \ldots, p_n)\, a^\dagger_{p_1} \cdots a^\dagger_{p_m} b^\dagger_{p_{m+1}} \cdots a^\dagger_{p_j} \cdots b^\dagger_{p_n} |0\rangle \\ &= \sum_{p_1,\ldots,p_n} \left[\mathbf{1}_n - \sum_{j=m+2}^{n} \widehat{T}_{m+1,j} \right] \chi(p_1, \ldots, p_n)\, a^\dagger_{p_1} \cdots a^\dagger_{p_{m+1}} b^\dagger_{p_{m+2}} b^\dagger_{p_n} |0\rangle \\ &= (n-m) \sum_{p_1,\ldots,p_n} \chi(p_1, \ldots, p_n)\, a^\dagger_{p_1} \cdots a^\dagger_{p_{m+1}} b^\dagger_{p_{m+2}} b^\dagger_{p_n} |0\rangle; \end{aligned} \tag{2.65}$$

in the last step we made use of the fact that χ is completely antisymmetric under the exchange of momenta p_i with i greater than m. According to equation (2.53), $C^a_+ |\psi\rangle$ is not zero if and only if

$$\chi_1(p_1, \ldots, p_n) \equiv A_n(1, \ldots, m+1)\, A_n(m+1, \ldots, n)\, \psi(p_1, \ldots, p_n) \neq 0; \tag{2.66}$$

the antisymmetrizers $A_n(m+2, \ldots, n)$ and $A_n(1, \ldots, m)$ have been omitted according to equation (2.8). Again, with the help of equation (2.52) we can write

$$C^a_+ |\psi\rangle = (n-m) \sum_{p_1,\ldots,p_n} \chi_1(p_1, \ldots, p_n)\, a^\dagger_{p_1} \cdots a^\dagger_{p_{m+1}} b^\dagger_{p_{m+2}} \cdots b^\dagger_{p_n} |0\rangle, \tag{2.67}$$

Aplying the operator C^a_+ several times we get the general result

$$(C^a_+)^j |\psi\rangle = \frac{(n-m)!}{(n-m-j)!} \sum_{p_1,\ldots,p_n} \chi_j(p_1, \ldots, p_n)\, a^\dagger_{p_1} \cdots a^\dagger_{p_{m+j}} b^\dagger_{p_{m+j+1}} \cdots b^\dagger_{p_n} |0\rangle, \tag{2.68}$$

with χ_j having been defined by

$$\chi_j(p_1, \ldots, p_n) \equiv A_n(1, \ldots, m+j)\, A_n(m+1, \ldots, n)\, \psi(p_1, \ldots, p_n); \tag{2.69}$$

therefore, the two conditions of the theorem, equations (2.57) and (2.58), are equivalent to

$$\begin{cases} (C^a_+)^{l-m} |\psi\rangle \neq 0, \\ (C^a_+)^{l-m+1} |\psi\rangle = 0, \end{cases} \tag{2.70}$$

which concludes the proof.

2.3.3 Summary

The results obtained in this section can be summarized in the following way: If the system has color and translational invariance, its eigenstates have the form

$$|Q,c,e\rangle = \frac{1}{\sqrt{m!}\sqrt{(n-m)!}} \sum_{p_1,\ldots,p_n} \psi_e(p_1,\ldots,p_n)\, a^\dagger_{p_1}\cdots a^\dagger_{p_m} b^\dagger_{p_{m+1}}\cdots b^\dagger_{p_n}\,|0\rangle; \qquad (2.71)$$

here the quantum numbers that label the states are: an integer Q which is the total momentum, the color c, the number of particles n, the number m which is related to the eigenvalue of C_3^a through the relation $c_3 = m - n/2$, and e which stands for any other quantum numbers that will be required to specify the energy level.

The color number c can be equal to $|\, m - n/2\,|$, $|\, m - n/2\,| + 1, \ldots, n/2$, and the wave function must satisfy conditions of equations (2.57) and (2.58), where $l = c + n/2$. The normalization constant β, and the factorials in the denominator are simply a matter of convention at this point. The wave function ψ_e vanishes, unless the sum of its arguments equals Q; and the eigenstate $|Q+q,e,c\rangle$ is obtained from equation (2.71) simply by replacing the wave function by

$$\psi_e(p_1 - q/n, \ldots, p_n - q/n).$$

2.4 Matrix elements of one-body operators

In this section we will derive general expressions for the matrix elements of one–body operators; two particular cases will be studied which will lead to the momentum distribution and the form factor. We will consider the one–dimensional system described in sections 2.2 and 2.3, whose eigenstates have the general form of equation (2.71), and our treatment will be restricted to operators which do not change the color of the system. One–body operators of that type have the general form

$$\Gamma = \sum_{p,q} \gamma(p,q)\,(a^\dagger_{p+q} a_p + b^\dagger_{p+q} b_p); \qquad (2.72)$$

when the operator $a^\dagger_{p+q}a_p$ acts on the states of the system, equation (2.71), the result is

$$a^\dagger_{p+q}a_p\,|Q,c,i\rangle = \frac{1}{\sqrt{m!}\sqrt{(n-m)!}}\sum_{p_1,\ldots,p_n}\psi_i(p_1,\ldots,p_n) \times\sum_{j=1}^{m}\delta_{p,p_j}\,a^\dagger_{p_1}\cdots a^\dagger_{p_j+q}\cdots a^\dagger_{p_m}b^\dagger_{p_{m+1}}\cdots b^\dagger_{p_n}\,|0\rangle; \tag{2.73}$$

in each term of the sum over j a factor of q can be substracted from the dummy index p_j, which will make the creation operators equal for any j

$$a^\dagger_{p+q}a_p\,|Q,c,i\rangle = \frac{1}{\sqrt{m!}\sqrt{(n-m)!}}\sum_{p_1,\ldots,p_n} \left[\sum_{j=1}^{m}\delta_{p,p_j-q}\,\psi_i(p_1,\ldots,p_j-q,\ldots,p_n)\right] a^\dagger_{p_1}\cdots a^\dagger_{p_m}b^\dagger_{p_{m+1}}\cdots b^\dagger_{p_n}\,|0\rangle. \tag{2.74}$$

The result for $b^\dagger_{p+q}b_p$ is similar, but the sum over j runs from $m+1$ to n; then, the operator Γ acting on the state $|Q,c,i\rangle$ gives

$$\Gamma\,|Q,c,i\rangle = \frac{1}{\sqrt{m!}\sqrt{(n-m)!}}\sum_{p_1,\ldots,p_n} \left[\sum_{q}\sum_{j=1}^{n}\gamma(p_j-q,q)\,\psi_i(p_1,\ldots,p_j-q,\ldots,p_n)\right] a^\dagger_{p_1}\cdots a^\dagger_{p_m}b^\dagger_{p_{m+1}}\cdots b^\dagger_{p_n}\,|0\rangle; \tag{2.75}$$

and if the states have been normalized, then the matrix elements of the operator Γ are

$$\langle Q+q,c,f\,|\,\Gamma\,|Q,c,i\rangle = \frac{1}{m!(n-m)!}\sum_{p_1,\ldots,p_n}\sum_{r_1,\ldots,r_n}\left[\sum_{j=1}^{n}\gamma(p_j-q,q)\,\psi_i(p_1,\ldots,p_j-q,\ldots,p_n)\right] \psi_f(r_1,\ldots,r_n)\,\langle 0|\,b_{r_n}\cdots b_{r_{m+1}}a_{r_m}\cdots a_{r_1}a^\dagger_{p_1}\cdots a^\dagger_{p_m}b^\dagger_{p_{m+1}}\cdots b^\dagger_{p_n}\,|0\rangle; \tag{2.76}$$

notice that there is no sum over q, because only if $q = Q' - Q$ the matrix element is different from zero. Using the commutation relations among the field operators, it is easy to show that

$$\langle 0|\,b_{r_n}\cdots b_{r_{m+1}}a_{r_m}\cdots a_{r_1}a^\dagger_{p_1}\cdots a^\dagger_{p_m}b^\dagger_{p_{m+1}}\cdots b^\dagger_{p_n}\,|0\rangle = \widetilde{\sum_{\nu\in S_n}}(-1)^\nu\left[\prod_{j=1}^{n}\delta(r_j-p_{\nu_j})\right], \tag{2.77}$$

where the sum runs over the $m!(n-m)!$ permutations of S_n such that $\nu_j \le j$ if $j \le m$. This result can be used to eliminate the sums over indices r_i in the matrix element of Γ, leading

to

$$\langle Q+q,c,f\,|\,\Gamma\,|\,Q,c,i\rangle = \frac{1}{m!(n-m)!}\sum_{p_1,\ldots,p_n} \left[\widetilde{\sum_{\nu\in S_n}}(-1)^\nu\,\psi_f(p_{\nu_1},\ldots,p_{\nu_n})\right]\times\left[\sum_{j=1}^{n}\gamma(p_j-q,q)\,\psi_i(p_1,\ldots,p_j-q,\ldots,p_n)\right]; \tag{2.78}$$

the term inside the first square brackets is the function ψ_f antisymmetrized on the first m arguments and the last $n-m$; namely, it has the form of χ_m defined by equation (2.54); thus, using the notation χ_m^f and adding a factor of q to the dummy index p_j, the matrix element can be written in the form

$$\langle Q+q,c,f\,|\,\Gamma\,|\,Q,c,i\rangle = \sum_{p_1,\ldots,p_n}\psi_i(p_1,\ldots,p_n)\sum_{j=1}^{n}\gamma(p_j,q)\,\chi_m^f(p_1,\ldots,p_j+q,\ldots,p_n). \tag{2.79}$$

Here χ_m^f is a sum of $m!(n-m)!$ functions ψ_f with their arguments permuted in some way; therefore, the matrix element of Γ has $m!(n-m)!$ terms of the form $\psi_i\psi_f$ times the function γ. However, since the states are eigenstates of C^2, the wave functions have some symmetries that can be used to reduce the number of terms.

There are two particular cases that are of interest to us, and which will be the subject of the next two subsections.

2.4.1 Momentum distribution

If the operator Γ does not change the total momentum of the system, $\gamma(p,q)$ must be of the form $\delta_{q,0}\gamma(p)$, and the operator becomes

$$\Gamma_0 = \sum_p \gamma(p)\,\widehat{N}(p), \tag{2.80}$$

where $\widehat{N}(p)$ is the momentum density operator defined in page 14. An example of this kind of operator is the kinetic energy; in general, this type of operator measures some intrinsic property of the system. The expectation value of Γ_0 is

$$\langle Q,c,i|\,\Gamma_0\,|Q,c,i\rangle = n\sum_p \gamma(p)\,N(p), \tag{2.81}$$

where $N(q)$ is the momentum distribution, equation (2.36). From equation (2.79), with $\gamma(p,q) = \delta_{q,0}\gamma(p)$, and $f = i$ we obtain the result

$$\begin{aligned} \langle Q,c,i|\,\Gamma\,|Q,c,i\rangle &= \sum_{p_1,\ldots,p_n} \psi(p_1,\ldots,p_n)\,\chi_m(p_1,\ldots,p_n)\sum_{j=1}^{n}\gamma(p_j), \\ &= \sum_{p_1,\ldots,p_n} \gamma(p_n)\sum_{j=1}^{n}\psi(p_1,\ldots,p_n,\ldots,p_j)\,\chi_m(p_1,\ldots,p_n,\ldots,p_j); \end{aligned} \tag{2.82}$$

the functions ψ and χ_m on the last line have the arguments p_j and p_n transposed; comparing this last result with equation (2.81), the functional form of the momentum distribution can be extracted

$$N(p_n) = \frac{1}{n}\sum_{p_1,\ldots,p_{n-1}}\sum_{j=1}^{n}\psi(p_1,\ldots,p_n,\ldots,p_j)\,\chi_m(p_1,\ldots,p_n,\ldots,p_j). \tag{2.83}$$

We can now calculate the norm of the states. If γ is replaced by one, the operator Γ becomes the number operator. In that case the expectation value of equation (2.82) becomes n times the square of the norm; we then have

$$\langle Q,c,i|\,Q,c,i\rangle = \sum_{p_1,\ldots,p_n}\psi(p_1,\ldots,p_n)\,\chi_m(p_1,\ldots,p_n). \tag{2.84}$$

2.4.2 Form factor

The second type of one–body operators that we will discuss are those such that $\gamma(p,q) = \gamma(q)$. They can be written in the form

$$\Gamma = \sum_q \gamma(q)\,\widehat{\rho}(q), \tag{2.85}$$

where $\widehat{\rho}(q)$ is the momentum-representation of the one-particle density operator (page 13). The matrix elements of Γ become

$$\langle Q+q,c,f|\,\Gamma_0\,|Q,c,i\rangle = n\,\gamma(q)\,F_{f,i}(q), \tag{2.86}$$

where $F_{f,i}$ is the inelastic form factor, equation (2.29). Comparing equations (2.86) and (2.79) we have

$$F_{f,i}(q) = \frac{1}{n}\sum_{p_1,\ldots,p_n}\psi_i(p_1,\ldots,p_n)\sum_{j=1}^{n}\chi_m^f(p_1,\ldots,p_j+q,\ldots,p_n). \tag{2.87}$$

The elastic form factor is obtained when $f = i$; however in that case the initial and final states are not completely the same, because the total momentum on the final state equals the initial one plus q. According to the discussion following equation (2.71), the wave function of the final state is the same as the initial one, except for a factor of q/n that must be substracted from all of the arguments; therefore, the elastic form factor is

$$F(q) = \frac{1}{n} \sum_{p_1,\ldots,p_n} \psi(p_1,\ldots,p_n) \sum_{j=1}^{n} \chi_m(p_1 - q/n,\ldots,p_j + q - q/n,\ldots,p_n - q/n); \qquad (2.88)$$

it can be verified that $F(0)$ equals the square of the norm, equation (2.84), and therefore if the state is normalized, then $F(0) = 1$.

2.5 Expectation value of two–body operators

We will now calculate the expectation value of two–body operators; the general form of color-invariant, two-body operators is

$$\Theta = \sum_{p,r,q} \sum_{\alpha,\beta} \theta(q)\, A^{\dagger}_{p-q,\alpha} A^{\dagger}_{r+q,\beta} A_{r,\beta} A_{p,\alpha} = \sum_{q} \theta(q)\, \widehat{P}(q), \qquad (2.89)$$

where $\widehat{P}(q)$ is the two-particle correlation operator, equation (2.34). An example of such an operator is the potential energy for a system of particles with two body interaction. The function $\theta(q)$ can be assumed to be even, without any loss of generality, because any general function is the sum of an even function plus an odd function, and if θ is odd then Θ becomes zero. The expectation value of the operator Θ can be written in the form

$$\langle \Theta \rangle = n(n-1) \sum_{q} \theta(q)\, P(q), \qquad (2.90)$$

where $P(q)$ is the two–particle correlation function defined by equation (2.32). In the two color case that we have been considering in the previous sections, the operators A should be replaced by either a or b, and the same for creation operators

$$\sum_{\alpha,\beta} A^{\dagger}_{p+q,\alpha} A^{\dagger}_{r-q,\beta} A_{r,\beta} A_{p,\alpha} = \qquad (2.91)$$

$$a^{\dagger}_{p+q} a^{\dagger}_{r-q} a_r a_p + b^{\dagger}_{p+q} b^{\dagger}_{r-q} b_r b_p + a^{\dagger}_{p+q} b^{\dagger}_{r-q} b_r a_p + b^{\dagger}_{p+q} a^{\dagger}_{r-q} a_r b_p;$$

then the expectation value of Θ can be calculated in the following way

$$\langle\Theta\rangle = \sum_q \theta(q)\left[\beta_c\, C(q) + 2\beta_d\, D(q)\right], \tag{2.92}$$

where $C(q)$ is the *same-color correlation function* and $D(q)$ the *different-color correlation function* defined by

$$C(q) \equiv \frac{1}{\beta_c}\left\langle \sum_{p,r}\left[a^\dagger_{p+q}a^\dagger_{r-q}a_r a_p + b^\dagger_{p+q}b^\dagger_{r-q}b_r b_p\right]\right\rangle, \tag{2.93}$$

$$D(q) \equiv \frac{1}{\beta_d}\left\langle \sum_{p,r} a^\dagger_{p+q}b^\dagger_{r-q}b_r a_p\right\rangle; \tag{2.94}$$

notice that the last two terms of equation (2.91) lead to the same result in the calculation of $\langle\Theta\rangle$ because θ is an even function. The normalization constants β_c and β_d are to ensure that the two correlation functions become 1 at $q = 0$, and thus they are

$$\beta_c \equiv n(n-1) - 2m(n-m), \tag{2.95}$$

$$\beta_d \equiv m(n-m). \tag{2.96}$$

The Fourier transforms of $C(q)$ and $D(q)$ give the probability of finding two particles, of the same or different color respectively, at some given distance from each other. To calculate $D(q)$, we aply the operator of equation (2.94) to the eigenstates of the system

$$a^\dagger_{p+q}b^\dagger_{r-q}b_r a_p\,|Q,c,i\rangle = \frac{1}{\sqrt{m!}\sqrt{(n-m)!}}\sum_{p_1,\ldots,p_n}\psi_i(p_1,\ldots,p_n) \tag{2.97}$$
$$\times\sum_{j=1}^{m}\sum_{l=m+1}^{n}\delta_{p,p_j}\,\delta_{r,p_l}\,a^\dagger_{p_1}\cdots a^\dagger_{p_j+q}\cdots a^\dagger_{p_m}b^\dagger_{p_{m+1}}\cdots b^\dagger_{p_l-q}\cdots b^\dagger_{p_n}\,|0\rangle;$$

which is very similar to equation (2.73); therefore, carrying out the same steps which lead to equation (2.79) we obtain the result

$$D(q) = \frac{1}{\beta_d}\sum_{p_1,\ldots,p_n}\psi(p_1,\ldots,p_n)\sum_{j=1}^{m}\sum_{l=m+1}^{n}\chi_m(p_1,\ldots,p_j+q,\ldots,p_l-q,\ldots,p_n); \tag{2.98}$$

a factor of q is added and substracted to the j'th and l'th arguments in χ_m .

It is easy to see that the result for $C(q)$ will be similar to that for $D(q)$, except that the momentum q is added and substracted to arguments which are both within the first m

positions or in the last $n-m$ places; hence we have

$$C(q) = \frac{1}{\beta_c} \sum_{p_1,\ldots,p_n} \psi(p_1,\ldots,p_n) \left[\sum_{1\le j<l\le m} \chi_m(p_1,\ldots,p_j+q,\ldots,p_l-q,\ldots,p_n) \right. \tag{2.99}$$
$$\left. + \sum_{m+1\le j<l\le n} \chi_m(p_1,\ldots,p_j+q,\ldots,p_l-q,\ldots,p_n) \right].$$

The momentum distribution, form factor and correlation functions contain information about the structure of the system. They can be used to calculate matrix elements and expectation values of one and two-body operators. These functions can all be measured in certain scattering processes; that will be the subject of the next chapter.

Chapter 3

Scattering theory in momentum representation

We will now explain how the dynamical functions that we are interested in (momentum distribution, form factors and correlation functions) are related to scattering experiments. Some results from the theory of scattering will be derived, using momentum representation. We are interested in non-relativistic electron scattering from nuclei; if the target nucleus is not too heavy, the first order Born approximation is expected to be valid. Our discussion has been extracted mainly from a review paper by West [30]. We will not discuss how to obtain empirical information about the quark momentum distribution; it is usually done by deep inelastic scattering which is a highly relativistic process and falls beyond the scope of this chapter.

The impulse approximation is also studied in this chapter. The term "impulse" originally referred to very abrupt and impulsive scattering processes; in those cases, the struck particle inside the target would be assumed to be free during the short time of the process [1]. As a result, the many-body problem would reduce to a two-body problem. We will use the term impulse approximation to refer to the general assumption that the intrinsic structure of the constituents of the target is not changed by their interaction.

[1] See for example chapter 12 in reference [31].

3.1 Target without internal structure

Consider two different kinds of elementary particles. One kind (*target* particles) which are created and annihilated by the operators $A^\dagger_{\mathbf{p},\beta}$ and $A_{\mathbf{p},\beta}$, which have simple fermionic commutation relations [2]

$$\{A_{\mathbf{p},\beta}, A^\dagger_{\mathbf{p}',\beta'}\} = \delta_{\beta,\beta'}\,\delta(\mathbf{p}-\mathbf{p}'). \tag{3.1}$$

The other kind of particles (*projectile* particles) are created and annihilated by the operators $\Psi^\dagger_{\mathbf{k},\alpha}$ and $\Psi_{\mathbf{k},\alpha}$, which are also assumed to be fermionic

$$\{\Psi_{\mathbf{k},\alpha}, \Psi^\dagger_{\mathbf{k}',\alpha'}\} = \delta_{\alpha,\alpha'}\,\delta(\mathbf{k}-\mathbf{k}'); \tag{3.2}$$

target and projectile operators commute with each other. We will assume that the two kinds of particles have a mutual interaction of the two–body type, and dependent on the relative distance (we will not consider interactions dependent upon the internal numbers α and β); in the language that we are using, that means that the interaction Hamiltonian has the form

$$H^I \equiv \sum_{\alpha,\beta}\int d\mathbf{p}\,d\mathbf{k}\,d\mathbf{q}\;v(\mathbf{q})\;\Psi^\dagger_{\mathbf{k}-\mathbf{q},\alpha}A^\dagger_{\mathbf{p}+\mathbf{q},\beta}A_{\mathbf{p},\beta}\Psi_{\mathbf{k},\alpha}. \tag{3.3}$$

Consider a system composed of one target particle which is not moving and a projectile that approaches the target with some initial momentum $\mathbf{k}$ and after being scattered, moves away from the target with a momentum $\mathbf{k}'$. A very large period of time before the collision, we assume that the interaction between projectile and target is zero; therefore the state of the system is

$$|i\rangle = \Psi^\dagger_{\mathbf{k},\alpha}A^\dagger_{\mathbf{0},\beta}\,|0\rangle; \tag{3.4}$$

as time goes on, the projectile approaches the target and the interaction is *adiabatically* switched on; a very large period of time after the collision, the projectile and target move freely, away from the point where the target was initially; the final state of the system is then

$$|f\rangle = \Psi^\dagger_{\mathbf{k}',\alpha'}A^\dagger_{\mathbf{q},\beta'}\,|0\rangle. \tag{3.5}$$

[2] Unlike the previous sections, we will now consider the general case in which momenta are continuous and three dimensional.

With the choice of commutation relation of equation (3.2), the wave function of a free projectile–like particle is

$$\langle \mathbf{r} |\, \Psi^{\dagger}_{\mathbf{k},\alpha} \,| 0 \rangle = (2\pi)^{-3/2}\, \chi_{\alpha}\, e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{3.6}$$

where χ is some representation of the internal symmetry group (spinor, bispinor, etc.); with this normalization, the scattering cross section for the process described above is given by [32, page 219]

$$\frac{d\sigma}{d\Omega} = (2\pi)^4\, m^2\, \frac{k'}{k}\; |\, \langle f |\, H^I \,| i \rangle \,|^2, \tag{3.7}$$

if the energy of the initial state equals that of the final state, and zero otherwise; m is the mass of the projectile. The matrix element of the interaction Hamiltonian is

$$\begin{aligned} &\langle f |\, H^I \,| i \rangle \sum_{\mu_1,\mu_2} \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_3\; v(\mathbf{p}_3) \\ &\quad \times \langle 0 |\, A_{\mathbf{q},\beta'} \Psi_{\mathbf{k}',\alpha'} \Psi^{\dagger}_{\mathbf{p}_1-\mathbf{p}_3,\mu_1} A^{\dagger}_{\mathbf{p}_2+\mathbf{p}_3,\mu_2} A_{\mathbf{p}_2,\mu_2} \Psi_{\mathbf{p}_1,\mu_1} \Psi^{\dagger}_{\mathbf{k},\alpha} A^{\dagger}_{\mathbf{0},\beta} \,| 0 \rangle; \end{aligned} \tag{3.8}$$

the expectation value of the operator on the second line becomes

$$\begin{aligned} &\langle 0 |\, A_{\mathbf{q},\beta'} A^{\dagger}_{\mathbf{p}_2+\mathbf{p}_3,\mu_2} A_{\mathbf{p}_2,\mu_2} A^{\dagger}_{\mathbf{0},\beta} \,| 0 \rangle \langle 0 |\, \Psi_{\mathbf{k}',\alpha'} \Psi^{\dagger}_{\mathbf{p}_1-\mathbf{p}_3,\mu_1} \Psi_{\mathbf{p}_1,\mu_1} \Psi^{\dagger}_{\mathbf{k},\alpha} \,| 0 \rangle \\ &\qquad = \delta_{\alpha,\alpha'}\, \delta_{\beta,\beta'}\, \delta_{\mu_2,\beta}\, \delta_{\mu_1,\alpha}\, \delta(\mathbf{p}_2)\, \delta(\mathbf{p}_3 - \mathbf{q})\, \delta(\mathbf{p}_1 - \mathbf{k})\, \delta(\mathbf{p}_3 + \mathbf{k}' - \mathbf{k}). \end{aligned} \tag{3.9}$$

With this last result, the matrix element of the interaction Hamiltonian becomes $v(\mathbf{q})$, and the scattering cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{pt} = (2\pi)^4\, m^2\, \frac{k'}{k}\, v^2(\mathbf{q}); \tag{3.10}$$

the label pt is to emphasize that the target is a point particle. With the normalization that we are using, $v(\mathbf{q})$ is the Fourier transform of the potential $v(\mathbf{r})$, divided by $(2\pi)^3$. For a given value of the initial momentum of the projectile $\mathbf{k}$, $\mathbf{q}$ and $\mathbf{k}'$ are completely specified by the magnitude of the momentum transfer q, because the three momenta must satisfy the conditions

$$\mathbf{q} = \mathbf{k} - \mathbf{k}', \tag{3.11}$$

$$\frac{q^2}{2M} = \frac{1}{2m}\left(k^2 - k'^2\right); \tag{3.12}$$

thus, if the potential is isotropic, the scattering cross section is a function of q only.

3.2 Composite target

Let us consider now the case when the target is no longer a point particle but an object made up of n point particles; the states of the target are now labeled by the total momentum $\mathbf{p}$, the total internal quantum number β (for example total spin, isospin, etc.), and a set of quantum numbers e that label the energy levels of the target; the target states are then of the form

$$|\mathbf{p},\beta,e\rangle = \Phi^\dagger_{\mathbf{p},\beta,e}\,|0\rangle, \tag{3.13}$$

where the operators $\Phi^\dagger$ are a superposition of elementary operators [3]

$$\begin{aligned}\Phi^\dagger_{\mathbf{p},\beta,e} = &\int d\mathbf{p}_1 \cdots \int d\mathbf{p}_n\, \delta(\mathbf{p}_1 + \cdots + \mathbf{p}_n - \mathbf{p}) \\ &\times\omega_e(\mathbf{p}_1 - \mathbf{p}/n, \ldots, \mathbf{p}_n - \mathbf{p}/n)\, A^\dagger_{\mathbf{p}_1,\beta_1} \cdots A^\dagger_{\mathbf{p}_n,\beta_n}.\end{aligned} \tag{3.14}$$

It has been assumed that the target system is translationally invariant; also, the wave function ω_e must have the right symmetries that will make the state an irreducible representation of the internal symmetry group, labeled by β. The energy of the state $|\,e,\beta,\mathbf{p}\rangle$ is $E_{e,\beta}$. The initial and final states of the target–projectile system are

$$|i\rangle = \Psi^\dagger_{\mathbf{k},\alpha}\,\Phi^\dagger_{\mathbf{0},\beta,i}\,|0\rangle, \tag{3.15}$$

$$|f\rangle = \Psi^\dagger_{\mathbf{k}',\alpha'}\,\Phi^\dagger_{\mathbf{q},\beta',f}\,|0\rangle; \tag{3.16}$$

and the matrix element of the interaction Hamiltonian is

$$\begin{aligned}\langle f|\,H^I\,|i\rangle &= \sum_{\mu_1,\mu_2}\int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_3\, v(\mathbf{p}_3) \\ &\quad\times\langle 0|\,\Phi_{\mathbf{q},\beta',f}\Psi_{\mathbf{k}',\alpha'}\Psi^\dagger_{\mathbf{p}_1-\mathbf{p}_3,\mu_1}A^\dagger_{\mathbf{p}_2+\mathbf{p}_3,\mu_2}A_{\mathbf{p}_2,\mu_2}\Psi_{\mathbf{p}_1,\mu_1}\Psi^\dagger_{\mathbf{k},\alpha}\Phi^\dagger_{\mathbf{0},\beta,i}\,|0\rangle, \\ &= \delta_{\alpha,\alpha'}\,v(\mathbf{k}-\mathbf{k}')\,\langle 0|\,\Phi_{\mathbf{q},\beta',f}\left[\sum_{\mu_2}\int d\mathbf{p}_2 A^\dagger_{\mathbf{p}_2+\mathbf{k}-\mathbf{k}',\mu_2}A_{\mathbf{p}_2,\mu_2}\right]\Phi^\dagger_{\mathbf{0},\beta,i}\,|0\rangle;\end{aligned} \tag{3.17}$$

the expectation value of the last line vanishes, unless $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ and $\beta = \beta'$. The scattering cross section becomes, from equation (3.7),

$$\left(\frac{d\sigma}{d\Omega}\right)_{i\to f} = n^2\left(\frac{d\sigma}{d\Omega}\right)_{pt}|\,F_{f,i}(\mathbf{q})\,|^2; \tag{3.18}$$

[3] This is a generalization of the result of equation (2.71).

where the scattering cross section $(d\sigma/d\Omega)_{pt}$ from a point target is given by equation (3.10), n is the number of particles in the target, and the *inelastic form factor* $F_{f,i}$, has been defined as in equation (2.29)

$$F_{f,i}(\mathbf{q}) \equiv \frac{1}{n}\langle \mathbf{q},\beta,f \mid \widehat{\rho}(\mathbf{q}) \mid \mathbf{0},\beta,i\rangle; \tag{3.19}$$

however the sum in the operator $\widehat{\rho}(\mathbf{q})$ of page 13 must be replaced by a three-dimensional integral

$$\widehat{\rho}(\mathbf{q}) = \sum_{\mu} \int d\mathbf{p} A^{\dagger}_{\mathbf{p}+\mathbf{q},\mu} A_{\mathbf{p},\mu}. \tag{3.20}$$

If the scattering is elastic, the internal state of the target does not change ($f = i$), and the cross section becomes

$$\frac{d\sigma}{d\Omega} = n^2 \left(\frac{d\sigma}{d\Omega}\right)_{pt} \mid F(\mathbf{q}) \mid^2, \tag{3.21}$$

where $F(\mathbf{q})$ is the *elastic form factor*, equation (2.28)

$$F(\mathbf{q}) \equiv \frac{1}{n}\langle \mathbf{q},\beta,i \mid \widehat{\rho}(\mathbf{q}) \mid \mathbf{0},\beta,i\rangle. \tag{3.22}$$

3.3 Inclusive scattering

The inelastic scattering cross section derived on the previous section, equation (3.18), is the cross section for some specific channel defined by the quantum numbers of the final state of the target: β', f. If no specific channels are selected the final state can include any of the allowed transitions, and equation (3.7) for the cross section must be modified by summing over all final states

$$\left(\frac{d^2\sigma}{d\Omega\, d\omega}\right)_{inc} = (2\pi)^4\, m^2\, \frac{k'}{k} \sum_{\beta',f} \mid \langle f \mid H^I \mid i\rangle \mid^2\ \delta(E - E'); \tag{3.23}$$

The label *inc* stands for *inclusive* scattering, and the energies E and E' inside the delta function are the initial and final energies of the system target–projectile

$$E = \frac{k^2}{2m} + E_{\beta,i}, \tag{3.24}$$

$$E' = \frac{k'^2}{2m} + \frac{q^2}{2M} + E_{\beta',f}, \tag{3.25}$$

where M is the mass of the target and m the mass of the projectile. The initial and final states have the same form as in the previous section, and the matrix element of the interaction Hamiltonian is given by equation (3.17); therefore, the inclusive scattering cross section becomes

$$\left(\frac{d^2\sigma}{d\Omega\, d\omega}\right)_{inc} = n^2 \left(\frac{d\sigma}{d\Omega}\right)_{pt} R(\mathbf{q},\omega), \tag{3.26}$$

where the *response function* $R(\mathbf{q},\omega)$ is defined by

$$R(\mathbf{q},\omega) \equiv \sum_f \delta\left(\frac{q^2}{2M} + E_{\beta,f} - E_{\beta,i} - \omega\right) \mid F_{f,i}(q) \mid^2; \tag{3.27}$$

here ω is the energy transferred to the target

$$\omega = \frac{1}{2m}(k^2 - k'^2). \tag{3.28}$$

The initial state is usually the ground state of the target; in that case $E_{\beta,i} \leq E_{\beta,f}$, where the equal sign is valid only when the final state is also the ground state (elastic channel). As a consequence, for a given momentum transfer $\mathbf{q}$, the energy transfer must be greater than or equal to the recoil energy

$$\omega \geq \frac{q^2}{2M}; \tag{3.29}$$

and the equal sign is attained only if the scattering process is elastic.

The sum over final states in the definition of the response function includes all states both bound and in the continuum part of the spectrum; in the continuum the sum over f must be replaced by an integral with some density of states. A theoretical calculation of the response function is then a complicated matter; however, sum rules are easier to predict. The *Coulomb sum rule* [33] is defined as the sum of the response function over all possible energy transfers ω, excluding the elastic channel

$$\Re(\mathbf{q}) \equiv \lim_{\epsilon\to 0^+} \int_{\frac{q^2}{2M}+\epsilon}^{\infty} d\omega\; R(\mathbf{q},\omega). \tag{3.30}$$

Using the definition of the response function, the integral is easily evaluated

$$\Re(\mathbf{q}) = \sum_{f\neq i} \mid F_{f,i}(\mathbf{q}) \mid^2 . \tag{3.31}$$

With the definition of the inelastic form factor, equation (3.19), we obtain the result

$$\begin{aligned}\Re(\mathbf{q}) &= \frac{1}{n^2}\sum_f \langle 0,\beta,i \mid \hat{\rho}^\dagger(\mathbf{q}) \mid \mathbf{q},\beta,f\rangle\langle \mathbf{q},\beta,f \mid \hat{\rho}(\mathbf{q}) \mid 0,\beta,i\rangle - \mid F(\mathbf{q}) \mid^2, \qquad (3.32)\\ &= \frac{1}{n^2}\sum_{\beta',f}\int d\mathbf{q}'\, \langle 0,\beta,i \mid \hat{\rho}^\dagger(\mathbf{q}) \mid \mathbf{q}',\beta',f\rangle\langle \mathbf{q}',\beta',f \mid \hat{\rho}(\mathbf{q}) \mid 0,\beta,i\rangle - \mid F(\mathbf{q}) \mid^2;\end{aligned}$$

in the last step we made use of the relation

$$\langle \mathbf{q}',\beta',f \mid \hat{\rho}(\mathbf{q}) \mid 0,\beta,i\rangle = \delta_{\beta,\beta'}\,\delta(\mathbf{q}-\mathbf{q}')\langle \mathbf{q}',\beta',f \mid \hat{\rho}(\mathbf{q}) \mid 0,\beta,i\rangle. \qquad (3.33)$$

The Coulomb sum rule can then be written as a correlation function

$$\Re(\mathbf{q}) = \frac{1}{n^2}\left[\langle\hat{\rho}^\dagger(\mathbf{q})\hat{\rho}(\mathbf{q})\rangle - \langle\hat{\rho}^\dagger(\mathbf{q})\rangle\langle\hat{\rho}(\mathbf{q})\rangle\right]; \qquad (3.34)$$

from the definition of $\hat{\rho}(\mathbf{q})$, equation (3.20), we find that

$$\hat{\rho}^\dagger(\mathbf{q})\,\hat{\rho}(\mathbf{q}) = \hat{N} + \hat{P}(\mathbf{q}), \qquad (3.35)$$

where $\hat{N}$ is the number operator, and $\hat{P}(\mathbf{q})$ the momentum-representation of the two-particle correlation operator. Hence, another way of writing the Coulomb sum rule is

$$\Re(\mathbf{q}) = \frac{1}{n}\left[1 + (n-1)\,P(\mathbf{q})\right] - \mid F(\mathbf{q}) \mid^2. \qquad (3.36)$$

In the normalization that we are using, the two-particle correlation function $P(\mathbf{q})$ equals one at $\mathbf{q} = 0$; and so does the elastic form factor. Thus, $\Re(\mathbf{q})$ is zero at $\mathbf{q} = 0$. At infinite momentum transfer, both $P(\mathbf{q})$ and $F(\mathbf{q})$ vanish, and the Coulomb sum rule goes asymptotically to $1/n$.

3.4 The impulse approximation

Protons and neutrons are not trully elementary, but made up by quarks. However, the dynamics of quarks inside the nucleons is a difficult problem. The underlying quark structure of the nucleons is usually acknowledged by introducing an intrinsic elastic form factor and a quark-momentum distribution which are empirically measured and assumed to be invariant. With this approximation, the problem of calculating quark-density functions of nuclear

targets can be solved. The problem of explaining theoretically the empirical properties of the nucleon is left to high energy physicists. Since one of the goals of this dissertation is to test the validity of that approximation, we will investigate in this section its implications regarding the functional form of the observables of the nucleus.

Since we are not considering isospin, we will treat protons and neutrons as the same particle (the nucleon). In the impulse approximation the internal structure of all the nucleons that form the nucleus is assumed to be the same, and identical to the structure of an isolated nucleon. The dynamics of the system is separated into two independent levels. Quark dynamics is used only to study an isolated nucleon; the result of that study gives the following intrinsic functions of nucleons:

$\rho(\mathbf{x})$ = density of quarks,

$n(\mathbf{p})$ = quark momentum distribution,

$f(\mathbf{q})$ = elastic form factor of the nucleon,

$p(\mathbf{x})$ = two-quark correlation (probability of finding two quarks a a distance $\mathbf{x}$ from each other).

On the other hand, an effective hadronic theory, in which the nucleons are considered elementary, is used to study the dynamics of the nucleus; the following functions are calculated:

$\rho_h(\mathbf{x})$ = density of nucleons,

$n_h(\mathbf{p})$ = momentum distribution of nucleons,

$f_h(\mathbf{q})$ = elastic form factor of the nucleus,

$p_h(\mathbf{x})$ = two-nucleon correlation;

the subindices h stand for *hadronic effective theory*. The quark observables of the nucleus are then obtained by combining the two levels.

3.4.1 Quark momentum distribution

The quark momentum distribution of the nucleus, $N(\mathbf{p})$, is the probability of finding a quark with momentum $\mathbf{p}$. If the quark which has momentum $\mathbf{p}$ is inside a nucleon which is at rest, the probability of such event is given by

$$N_o(\mathbf{p}) = n_h(\mathbf{0})\, n(\mathbf{p}). \tag{3.37}$$

If the nucleon is moving with momentum $\mathbf{q}$, its internal structure is still the same. The wave function of the moving nucleon should then have the Galilean invariant form of equation (2.38); therefore, its momentum distribution is $n(\mathbf{p} - \mathbf{q}/n_q)$, where n_q is the number of quarks inside the nucleon. The quark momentum distribution of the nucleus is then

$$N_{IA}(\mathbf{p}) = \int d\mathbf{q}\; n_h(\mathbf{q})\; n(\mathbf{p} - \mathbf{q}/n_q). \tag{3.38}$$

this result is referred to as the *convolution approximation* in the literature [9,14].

3.4.2 Elastic form factor

The probability of finding a nucleon at the position $\mathbf{R}$ is given by $\rho_h(\mathbf{R})$. And the probability of finding a quark in that nucleon, at a position $\mathbf{x}$ from the center of mass is $\rho(\mathbf{x})$. The position of the quark with respect to the nucleus would be $\mathbf{r} = \mathbf{R} + \mathbf{x}$. Hence, the quark density in the nucleus is

$$\rho_{IA}(\mathbf{r}) = \int d\mathbf{R}\; \rho_h(\mathbf{R})\; \rho(\mathbf{r} - \mathbf{R}). \tag{3.39}$$

The Fourier transform of this convolution integral gives the elastic form factor of the nucleus:

$$F_{IA}(\mathbf{q}) = f_h(\mathbf{q})\; f(\mathbf{q}). \tag{3.40}$$

3.4.3 Two-quark correlation function

Suppose there is a quark at a position $\mathbf{r}_1$ inside a nucleon which is at the position $\mathbf{R}_1$ in the nucleus. Another quark is at the position $\mathbf{r}_2$ inside another nucleon at $\mathbf{R}_2$ (figure 3-1). The probability for these two events is

$$\rho(\mathbf{r}_1)\; \rho(\mathbf{r}_2)\; p_h(\mathbf{R}), \tag{3.41}$$

where $\mathbf{R}$ is the position of the second nucleon with respect to the first one ($\mathbf{R} = \mathbf{R}_2 - \mathbf{R}_1$). The position of the second quark with respect to the first one is

$$\mathbf{r} = \mathbf{R} + \mathbf{r}_2 - \mathbf{r}_1. \tag{3.42}$$

The probability of finding two quarks —from different nucleons— at a relative distance $\mathbf{r}$ from each other is then

$$P_{dif}(\mathbf{r}) = \int\int d\mathbf{r}_1\; d\mathbf{r}_2\; \rho(\mathbf{r}_1)\; \rho(\mathbf{r}_2)\; p_h(\mathbf{r} + \mathbf{r}_1 - \mathbf{r}_2). \tag{3.43}$$

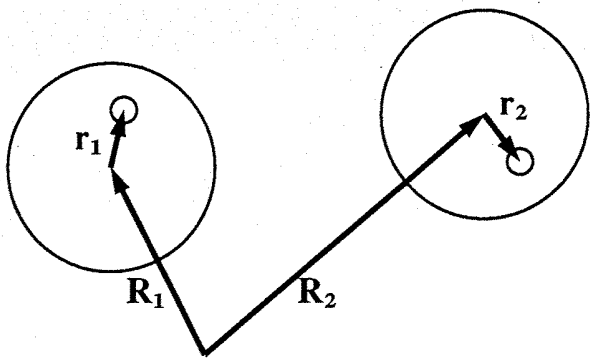


Figure 3-1: Two-quark correlation in the impulse approximation. There are two quarks at the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ inside two nucleons which are at the positions $\mathbf{R}_1$ and $\mathbf{R}_2$.

The two quarks can also come from the same nucleon, in which case

$$P_{same}(\mathbf{r}) = p(\mathbf{r}). \tag{3.44}$$

If there are n_q quarks inside each nucleon and a total of n in the whole nucleus, when two quarks are picked randomly the probability that they belong to the same nucleon is

$$\frac{n_q - 1}{n - 1}; \tag{3.45}$$

and the probability that they come from different nucleons is

$$\frac{n - n_q}{n - 1}. \tag{3.46}$$

The two-quark correlation of the nucleus is then

$$P_{IA}(\mathbf{r}) = \frac{1}{n-1}\left[(n_q - 1)\, p(\mathbf{r}) + (n - n_q) \int\int d\mathbf{r}_1\, d\mathbf{r}_2\, \rho(\mathbf{r}_1)\, \rho(\mathbf{r}_2)\, p_h(\mathbf{r} + \mathbf{r}_1 - \mathbf{r}_2)\right]. \tag{3.47}$$

And its Fourier transform is

$$\begin{aligned} P_{IA}(\mathbf{q}) &= \int d\mathbf{r}\, e^{i\,\mathbf{q}\cdot\mathbf{r}}\, P(\mathbf{r}) \\ &= \frac{1}{n-1}\left[(n_q - 1) \int d\mathbf{r}\, e^{i\mathbf{q}\cdot\mathbf{r}}\, p(\mathbf{r}) \right. \\ &\qquad \left. + (n - n_q) \int d\mathbf{r}_1\, e^{-i\mathbf{q}\cdot\mathbf{r}_1}\, \rho(\mathbf{r}_1) \int d\mathbf{r}_2\, e^{i\mathbf{q}\cdot\mathbf{r}_2}\, \rho(\mathbf{r}_1) \int d\mathbf{R}\; e^{i\mathbf{q}\cdot\mathbf{R}}\, p_h(\mathbf{R})\right] \\ &= \frac{1}{n-1}\left[(n_q - 1)\, p(\mathbf{q}) + (n - n_q)\, f(\mathbf{q})\, f(-\mathbf{q})\, p_h(\mathbf{q})\right]; \end{aligned} \tag{3.48}$$

if the density of quarks inside the nucleon is spherically symmetric, then $f(\mathbf{q}) = f(-\mathbf{q})$ and we have

$$P_{IA}(\mathbf{q}) = \frac{1}{n-1}\left[(n_q - 1)\, p(\mathbf{q}) + (n - n_q)\, f^2(\mathbf{q})\, p_h(\mathbf{q})\right]. \tag{3.49}$$

3.4.4 The closure approximation

In the impulse approximation the one-quark density operator is

$$\widehat{\rho}_{IA}(\mathbf{r}) = \int d\mathbf{R}\, \widehat{\rho}_h(\mathbf{R})\, \widehat{\rho}(\mathbf{r} - \mathbf{R}); \tag{3.50}$$

and its Fourier transform

$$\widehat{\rho}_{IA}(\mathbf{q}) = \widehat{\rho}_h(\mathbf{q})\, \widehat{\rho}(\mathbf{q}). \tag{3.51}$$

The inelastic form factor becomes

$$F_{f,i}(\mathbf{q}) = \frac{1}{n}\langle N_f \mid \widehat{\rho}(\mathbf{q}) \mid N_i\rangle\langle \psi_f \mid \widehat{\rho}_h(\mathbf{q}) \mid \psi_i\rangle, \tag{3.52}$$

where the states $|\, N\rangle$ and $|\, \psi\rangle$ are the quark state of the nucleon, and the state of the nucleus in the effective theory. If we assume that the initial and final states of the nucleon are the same, we can write the form factor as

$$F_{f,i}(\mathbf{q}) = \frac{f(\mathbf{q})}{n_h}\langle \psi_f \mid \widehat{\rho}_h(\mathbf{q}) \mid \psi_i\rangle; \tag{3.53}$$

here $f(\mathbf{q})$ is the elastic form factor of the nucleon, n_h is the number of hadrons, and the second term is the inelastic form factor of the nuclear system in the effective hadronic theory. This approximation is known as the "closure" approximation.

With this approximation, the Coulomb sum rule, equation (3.31), becomes

$$\Re_{IA}(\mathbf{q}) = \frac{f^2(\mathbf{q})}{n_h^2}\left[\langle \widehat{\rho}_h^{\dagger}(\mathbf{q})\, \widehat{\rho}_h(\mathbf{q})\rangle - \langle \widehat{\rho}_h(\mathbf{q})\rangle^2\right]; \tag{3.54}$$

the two expectation values inside the square brackets are related to the hadronic elastic form factor and hadron correlation function by the relations

$$\langle \widehat{\rho}_h(\mathbf{q})\rangle = n_h\, f_h(\mathbf{q}), \tag{3.55}$$

$$\langle \widehat{\rho}_h^{\dagger}(\mathbf{q})\, \widehat{\rho}_h(\mathbf{q})\rangle = n_h\, [1 + (n_h - 1)\, p_h(\mathbf{q})], \tag{3.56}$$

where f_h and p_h are the elastic form factor and correlation function of the nucleus in the effective theory. The Coulomb sum rule then becomes

$$\Re_{IA}(\mathbf{q}) = \frac{f^2(\mathbf{q})}{n_h}\left[1 + (n_h - 1)\, p_h(\mathbf{q})\right] - f^2(\mathbf{q})\, f_h^2(\mathbf{q}). \tag{3.57}$$

At zero momentum transfer the elastic form factors and the hadron correlation function are 1; then $\Re_{IA}(0) = 0$. In the limit when q goes to infinity f, f_h and p_h vanish and the Coulomb sum rule goes to zero and not to one over the number of quarks as would be the case in a quark theory of the nucleus.

Another important quantity that is usually defined [4], is the function

$$S(\mathbf{q}) \equiv \frac{n_h\, \Re_{IA}(\mathbf{q})}{f^2(\mathbf{q})}, \tag{3.58}$$

which goes to one in the asymptotic limit. In a real scattering experiment the domain of energy transfer at which the cross section is measured, is limited by the characteristics of the beam. Different peaks at various regions are identified as the result of distinct mechanisms: quasielastic scattering, meson production, etc. It is not possible to measure the general Coulomb sum rule $\Re(q)$; one had better try to identify the form that the sum rule would have as constrained by a specific mechanism. For quasielastic scattering, the "closure" approximation is expected to be valid, and the relevant sum rule is the function $S(q)$. The cross section for quasielastic scattering peaks at an energy transfer $\omega \approx q^2/2m$, where m is the mass of the nucleon; beyond that it falls rapidly. The asymptotic behavior far from the quasielastic peak can be inferred — neglecting the neighboring peaks which are due to different mechanisms— and $S(q)$ can be calculated from the empirical data. We will return to this point in chapters 7 and 8.

[4] See for example reference [30]

Chapter 4

Many–body system with delta–function interaction

The problem discussed in this chapter is a one–dimensional system of particles with delta–function interaction. The field-theoretical formulation of the problem is known as the *non-linear Schrödinger model.* It can be solved by introducing a finite volume with periodic boundary conditions; the Bethe *ansatz* method is then used, and the limit when the volume goes to infinity is taken. A more direct and general approach is the *inverse scattering method.* Only the Bethe *ansatz* approach is reviewed here, because in our model we will be interested in the case of finite volume.

We will focus our attention first on the case of bosons with no internal degrees of freedom, and then on the case of fermions with two *colors*; both cases will be used in chapter 5 to construct a model of the nucleus and approximation schemes. The two problems that will be studied have been solved since the 1960's; there is an extensive bibliography on the subject which can be found in the review papers of references [34,35,26].

An alternate form of the Bethe *ansatz* which has not received much attention in the literature is the momentum representation [36,22]. We will use this approach in the next chapters, because it makes the calculation of observables a tractable problem. The momentun representation of the Bethe *ansatz* has a smooth functional form in all regions of momentum space. That is not the case in configuration space: the first derivative of the wave function

has discontinuities across the boundaries of $n!$ regions, where n is the number of particles. This makes difficult even the calculation of the norm for a small number of particles; as the number of particles grows, the calculations become more intractable and a calculation of correlation functions for the general case has not been done.

We will introduce in chapter 6 a new technique to calculate correlation functions and other observables, using the momentum representation. A crucial factor in those calculations is the use of a simple form for the states, exploiting the symmetries explained in chapter 2. By the end of this chapter we will give that simple form in the case of a two-cluster system, which is an original result from our project.

4.1 The Bethe ansatz

In 1931 H. Bethe proposed a method to construct the eigenfunctions of the one-dimensional Heisenberg spin chain. Since then, eigenfunctions of the same type have been shown to give the solution to a diversity of integrable systems in condensed matter and quantum field theory [34,35,26].

In the context of the many-body system with delta-function interaction, the Bethe *ansatz* method was first introduced by Lieb and Liniger[37] in 1963. They treated the problem in first-quantization; the same approach was used later for the case of fermions [38,39]. We find it more convenient to use second-quantization, but we will make a connection with the first-quantization formalism because it gives a more intuitive idea of the system under consideration. Also, a very extensive study of Bethe *ansatz* models has been done in first-quantization.

4.2 Bosonic field without internal degrees of freedom

Within the last two decades, a few non–trivial integrable quantum systems have been studied in detail; that research has led to the development of the theory of quantum integrable systems. One of those systems is the *nonlinear Schrödinger model* [1]. It is described by the

[1] A general discussion of integrable models can be found in reference [40]

Hamiltonian

$$\widehat{H} = \int_{-\infty}^{+\infty} dx \, \left[\partial_x \phi^\dagger(x) \, \partial_x \phi(x) + c \, \phi^\dagger(x) \, \phi^\dagger(x) \, \phi(x) \, \phi(x)\right]; \tag{4.1}$$

the operators $\phi^\dagger(x)$ and $\phi(x)$ create and annihilate bosons at the point x; their commutator is

$$\left[\phi(x), \phi^\dagger(y)\right] = \delta(x-y). \tag{4.2}$$

The equation of motion derived from the Hamiltonian of equation (4.1) is the *nonlinear Schrödinger equation.* A state of n bosons has the general form

$$\mid \psi\rangle = \int dx_1 \cdots \int dx_n \, \psi(x_1, \ldots, x_n) \, \phi^\dagger(x_1) \cdots \phi^\dagger(x_n) \mid 0\rangle; \tag{4.3}$$

it is easy to see that the condition for $\mid \psi\rangle$ to be an eigenstate of the nonlinear Schrödinger Hamiltonian is that the wave function ψ be a solution to the equation

$$\left[-\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \le j < l \le n} \delta(x_j - x_l)\right] \chi(x_1, \ldots, x_n) = E \, \chi(x_1, \ldots, x_n), \tag{4.4}$$

where $\chi(x_1, \ldots, x_n)$ is obtained from $\psi(x_1, \ldots, x_n)$ by making it completely symmetric:

$$\chi(x_1, \ldots, x_n) = S_n(1, \ldots, n) \, \psi(x_1, \ldots, x_n); \tag{4.5}$$

the symmetrizer S_n has been defined in chapter 2. Equation (4.4) is the Schrödinger equation for a system of n bosons with an interaction of the form $g\delta(x_j - x_l)$, between all pairs of particles j and l, x_j being the position of particle j. The mass of the particles is m, and the constant c has been defined as $c = mg$; kinetic and potential energies are being measured in units of $1/2m$, namely, the eigenvalue E must be multiplied times $1/2m$ to convert it into natural units.

Such a many-body system has been solved in detail by Lieb and Liniger [37]. They have proved that the eigenfunctions χ have the form of the Bethe *ansatz*

$$\chi(x_1, \ldots, x_n) = \sum_{\mu, \nu \in S_n} a_\mu \, \theta(x_{\nu_1}, \ldots, x_{\nu_n}) \, \exp\left(i \sum_{j=1}^{n} k_{\mu_j} x_{\nu_j}\right), \tag{4.6}$$

where μ and ν are permutations of order n; the step function $\theta(x_{\nu_1},\ldots,x_{\nu_n})$ has been defined as [26, page 632]

$$\theta(x_{\nu_1},\ldots,x_{\nu_n}) \equiv \begin{cases} 1, \text{ if } x_{\nu_1} < x_{\nu_2} < \cdots < x_{\nu_n} \\ 0, \text{ otherwise} \end{cases} \tag{4.7}$$

the numbers $\{k_1,\ldots,k_n\}$ are called *Bethe momenta*, and they are the quantum numbers that label the eigenfunctions χ; their sum gives the total momentum of the system

$$Q = \sum_{j=1}^{n} k_j. \tag{4.8}$$

To evaluate the coefficients a_ν, we notice that from equation (4.4) it follows that the derivatives of χ must have certain discontinuities:

$$\lim_{\epsilon\to 0^+}\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_l}\right)\chi(x_1,\ldots,x_n)\,|_{x_j=x_l-\epsilon}^{x_j=x_l+\epsilon} = 2c\,\chi(x_1,\ldots,x_n)\,|_{x_j=x_l}, \tag{4.9}$$

for any pair of indices (j,l). These conditions imply that only one of the $n!$ coefficients a_μ is linearly independent; if the coefficient a_μ is chosen to be 1 when the permutation μ is the identity, then the general expression for a_μ is [2]

$$a_\mu = \prod_{l<j,\,(\mu_l>\mu_j)} \frac{k_{\mu_l} - k_{\mu_j} - ic}{k_{\mu_l} - k_{\mu_j} + ic}, \tag{4.10}$$

where, for a fixed permutation μ, the product runs only over those pairs (l,j) such that $\mu_l > \mu_j$. The wavefunctions are completely defined by the Bethe momenta $\{k_j\}$; the sum of their squares gives the eigenvalue of the Hamiltonian

$$\widehat{H} \mid \psi(k_1,\ldots,k_n)\rangle = \sum_{j=1}^{n} k_j^2 \mid \psi(k_1,\ldots,k_n)\rangle. \tag{4.11}$$

Equations (4.9) can be satisfied only if all of the Bethe momenta are different. Some of the momenta can be complex, if the interaction is attractive; in that case they have the same real part, but different imaginary parts. Except for the cases of complex momenta, which leads to bound states, the energy spectrum resembles that of a system of fermions, in spite of the bosonic nature of the field, due to the fact that any pair of momenta k_j must be different.

[2] The details of the derivation of a_μ are found in reference [37]

This is an important feature of Bethe *ansatz* models that we will use in chapter 5 where a nuclear model is constructed.

The wave function $\chi(x_1,\ldots,x_n)$, equation (4.6), is completely symmetric; however, when the states $\mid \psi\rangle$ are constructed according to equation (4.3), ψ does not have to be symmetric but it can be any function that when symmetrized leads to $\chi(x_1,\ldots,x_n)$; for instance, we can write $\mid \psi\rangle$ in the form

$$\mid \psi(k_1,\ldots,k_n)\rangle = \tag{4.12}$$
$$n! \int dx_1 \cdots \int dx_n \sum_{\mu\in S_n} a_\mu\, \theta(x_{\mu_1},\ldots,x_{\mu_n})\, \exp\left[i\sum_{j=1}^{n} k_j x_j\right]\, \phi^\dagger(x_1)\cdots\phi^\dagger(x_n)\mid 0\rangle.$$

4.2.1 Periodic boundary conditions

If the system of bosons is placed inside a box of length L, with periodic boundary conditions, periodicity conditions must be imposed on $\chi(x_1,\ldots,x_n)$ and its first derivatives

$$\chi(-L/2,x_2,\ldots,x_n) = \chi(L/2,x_2,\ldots,x_n), \tag{4.13}$$
$$\frac{\partial}{\partial x_1}\chi(x_1,x_2,\ldots,x_n)\mid_{x_1=-L/2} = \frac{\partial}{\partial x_1}\chi(x_1,x_2,\ldots,x_n)\mid_{x_1=L/2}; \tag{4.14}$$

periodicity on all of the other arguments follows, because χ is completely symmetric. This periodicity condition implies that the Bethe momenta are no longer arbitrary but they must be a solution to the system of equations [37]

$$\prod_{l=1,\,(l\neq j)}^{n} \frac{k_l - k_j - ic}{k_l - k_j + ic} = e^{ik_jL}, \quad j=1,\ldots,n; \tag{4.15}$$

these equations can be written in terms of inverse trigonometric functions; however, we must keep in mind that the inverse trigonometric functions have an infinite number of branches. We will use the inverse cotangent function; for points x on the positive real semiaxis, the following relation holds

$$\cot^{-1} x = \frac{1}{2i}\ln\left(\frac{x+i}{x-i}\right), \tag{4.16}$$

where, for complex numbers z

$$\ln z \equiv \ln \mid z \mid + i\theta, \tag{4.17}$$

and if the phase θ is defined within the interval $(-\pi, \pi]$, then the branch of $\cot^{-1} x$ that extends from $\pi/2$ to 0 is obtained. Equation (4.16) can be used to define the analytical continuation of the function $\cot^{-1}$ onto the complex plane

$$\cot^{-1} z = \frac{1}{2i} \ln\left(\frac{z+i}{z-i}\right); \tag{4.18}$$

this definition leads to a function which has a branch cut along the segment connecting the points $-i$ and i; [3] also

$$-\frac{\pi}{2} < Re(\cot^{-1} z) \leq \frac{\pi}{2}. \tag{4.19}$$

Taking the logarithm on both sides, and using equation (4.18), the system of equations (4.15) become

$$k_j = \frac{2\pi}{L} n_j + \sum_{l=1}^{n} k_{j,l}, \tag{4.20}$$

$$k_{j,l} \equiv \frac{2}{L} \cot^{-1}\left(\frac{k_j - k_l}{c}\right) \quad (j \neq l), \tag{4.21}$$

$$k_{j,j} \equiv 0; \tag{4.22}$$

the quantities $k_{j,l}$ are called *auxiliary Bethe momenta*; and the numbers n_j are any set of integers. There is an infinite number of solutions k_j, but for each given set of numbers $\{n_j\}$ there corresponds only one of them.

4.2.2 The Bethe ansatz in momentum representation

The derivatives of the wave functions of equation (4.6) are discontinuous at the boundaries of $n!$ regions of the configuration space $x_1, \ldots, x_n$; that makes the calculation of matrix elements of operators an intractable problem. In momentum space, the wave functions are continuous, as we will see on this subsection.

The introduction of periodic boundary conditions constrains the momentum domain. The allowed values of any physical momenta are integer multiples of $2\pi/L$. The Fourier series of

[3] If one wants to use a branch of $\cot^{-1}$ that is not discontinuous at the origin, one can choose to measure phases in the interval $[0, 2\pi)$, which leads to a branch cut along the points of the imaginary axis $\mid y \mid \geq i$. Such a branch is discontinuous at the point at infinity.

the field operators are obtained from equations (2.16) and (2.17), if the indices α are dropped

$$\phi^\dagger(x) = \frac{1}{\sqrt{L}} \sum_p \exp\left(-i\frac{2\pi}{L}px\right) A_p^\dagger, \tag{4.23}$$

$$\phi(x) = \frac{1}{\sqrt{L}} \sum_p \exp\left(i\frac{2\pi}{L}px\right) A_p, \tag{4.24}$$

where $A_p^\dagger$ and A_p are the operators that create and annihilate a particle with momentum $2\pi p/L$, and p is an integer. If we substitute the field operators in equation (4.1) by their Fourier series, then the nonlinear Schrödinger Hamiltonian becomes

$$\hat{H} = \left(\frac{2\pi}{L}\right)^2 \sum_p p^2 A_p^\dagger A_p + \frac{c}{L} \sum_{p,q,r} A_{p+q}^\dagger A_{r-q}^\dagger A_r A_p; \tag{4.25}$$

We will introduce dimensionless units: if all momenta are measured in units of $2\pi/L$, then the allowed momenta are integers p; energies, which hitherto have been measured in units of $1/2m$, will now be measured in units of $(2\pi/L)^2/2m$. With this change of units[4] the Hamiltonian takes the form

$$\hat{H} = \sum_p p^2 A_p^\dagger A_p + \frac{cL}{4\pi^2} \sum_{p,q,r} A_{p+r}^\dagger A_{q-r}^\dagger A_q A_p; \tag{4.26}$$

there is only one parameter in the model: the dimensionless quantity cL.

The eigenstates of this Hamiltonian are the momentum representation of the Bethe ansatz. Sasaki and Kebukawa have shown that they are [36, eqs. 2.19-20]

$$|\,\psi(n_1,\ldots,n_n)\rangle = \frac{\beta}{\sqrt{n!}} \sum_{p_{1,2},\ldots,p_{n-1,n}} \prod_{1\le l<j\le n} (p_{l,j} - k_{l,j})^{-1} A_{q_1}^\dagger \ldots A_{q_n}^\dagger\, |0\rangle, \tag{4.27}$$

$$q_j \equiv n_j + \sum_{l=1}^{n} p_{j,l} \qquad (p_{j,l} \equiv -p_{l,j}), \tag{4.28}$$

where the product is over all pairs of non–repeated indices (l,j) and the sum is over the integers $p_{i,m}$ with $1 \le i < m \le n$.

The integers n_j and the auxiliary momenta $k_{j,l}$ were introduced in equations (4.21) and (4.22), which in dimensionless units should be written in the form

$$k_j = n_j + \sum_{l=1}^{n} k_{j,l}, \tag{4.29}$$

[4]The dimensionless quantities that we deal with from now on must be multiplied times $2\pi/L$, in the case of momenta, and times $2\pi^2/mL^2$ in the case of energies, to convert them into natural units

$$k_{j,l} \equiv \frac{1}{\pi}\cot^{-1}\left[\frac{2\pi(k_j - k_l)}{cL}\right] \quad (j \neq l), \tag{4.30}$$

$$k_{j,j} \equiv 0; \tag{4.31}$$

according to equation (4.19), the real parts of $k_{i,j}$ are between -1/2 and 1/2.

Two-boson system

As a simple example, let us consider the case of only two bosons. There are two Bethe momenta k_1 and k_2, and only one independent auxiliary quantity $k_{1,2}$. From equations (4.8) and (4.29) (with $n = 2$), it follows that

$$k_1 + k_2 = n_1 + n_2 = Q, \tag{4.32}$$

where Q is the total momentum. If the integer l and the momentum k are defined as

$$l \equiv 2n_1 - Q = -2n_2 + Q, \tag{4.33}$$

$$k \equiv k_1 - \frac{Q}{2} = -k_2 + \frac{Q}{2}, \tag{4.34}$$

the Bethe equations (4.29) become

$$k = \frac{l}{2} + \frac{1}{\pi}\cot^{-1}\left(\frac{4\pi k}{cL}\right). \tag{4.35}$$

The states are obtained from equation (4.27), with $n = 2$

$$\begin{aligned} |\psi\rangle &= \frac{\beta}{\sqrt{2}}\sum_{p_{1,2}}(p_{1,2} - k_{1,2})^{-1}\, A^\dagger_{n_1+p_{1,2}} A^\dagger_{n_2-p_{1,2}}\,|0\rangle \\ &= \frac{\beta}{\sqrt{2}}\sum_{p_1,p_2}\frac{\delta(p_1+p_2-Q)}{p_1 - n_1 - k_{1,2}}\, A^\dagger_{p_1} A^\dagger_{p_2}\,|0\rangle; \end{aligned} \tag{4.36}$$

and in terms of the momentum k, it can be written in the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}\sum_{p_1,p_2}\psi(p_1 - Q/2, p_2 - Q/2)\, A^\dagger_{p_1} A^\dagger_{p_2}\,|0\rangle, \tag{4.37}$$

$$\psi(p_1,p_2) \equiv \frac{\beta\,\delta(p_1+p_2)}{p_1 - k}; \tag{4.38}$$

therefore, the internal structure of the two-boson system is determined by the momentum k, which depends on only one quantum number l. The quantum number is not completely arbitrary, because in its definition it was implicit the condition

$$l \pmod 2 = Q \pmod 2. \tag{4.39}$$

And l can be assumed to be positive, without any loss of generality.

4.3 Fermionic field with an internal degree of freedom

If the field has some internal symmetry, which as in chapter 2 we call *color*, the nonlinear Schrödinger Hamiltonian becomes

$$\widehat{H} = \int dx \left[\sum_{\alpha} \partial_x \phi_\alpha^\dagger(x)\, \partial_x \phi_\alpha(x) + c \sum_{\alpha,\beta} \phi_\alpha^\dagger(x)\, \phi_\beta^\dagger(x)\, \phi_\beta(x)\, \phi_\alpha(x) \right]; \tag{4.40}$$

it has been assumed that the interaction is color invariant. If the field is fermionic, then

$$\{\phi_\alpha(x), \phi_\beta^\dagger(y)\} = \delta_{\alpha,\beta}\, \delta(x-y), \tag{4.41}$$

$$\{\phi_\alpha(x), \phi_\beta(y)\} = \{\phi_\alpha^\dagger(x), \phi_\beta^\dagger(y)\} = 0; \tag{4.42}$$

and the color indices in the interaction term must be different, since two fermions with the same quantum numbers cannot be created. If the field operators are substituted by their Fourier series, equations (2.16) and (2.17), the Hamiltonian becomes

$$\widehat{H} = \left(\frac{2\pi}{L}\right)^2 \sum_{\alpha,p} p^2\, A_{\alpha,p}^\dagger A_{\alpha,p} + \frac{c}{L} \sum_{\alpha\neq\beta} \sum_{p,q,r} A_{\alpha,p+q}^\dagger A_{\beta,r-q}^\dagger A_{\beta,r} A_{\alpha,p}. \tag{4.43}$$

If there are only two colors (a and b), using the simplified notation defined in equation (2.40), $\widehat{H}$ can be written as

$$\widehat{H} = \left(\frac{2\pi}{L}\right)^2 \sum_{p} p^2\, (a_p^\dagger a_p + b_p^\dagger b_p) + \frac{2c}{L} \sum_{p,q,r} a_{p+q}^\dagger b_{r-q}^\dagger b_r a_p. \tag{4.44}$$

Let us now look at the eigenvalue problem

$$\widehat{H}\ |\psi\rangle = E\ |\psi\rangle; \tag{4.45}$$

since $\widehat{H}$ commutes with the number operator, and with the color operators C_3^a and C^2, equations (2.43) and (2.46), $|\psi\rangle$ must have the general form of equation (2.71)

$$|\psi\rangle = \sum_{p_1,\cdots,p_n} \psi(p_1,\ldots,p_n)\, a^\dagger_{p_1}\cdots a^\dagger_{p_m} b^\dagger_{p_{m+1}} b^\dagger_{p_n}\,|0\rangle. \tag{4.46}$$

For a given number of particles, n, the color of the states $|\psi\rangle$ can take any value $c = l - n/2$ (theorem 1, page 18), where l must be an integer in the interval

$$\frac{n}{2} \le l \le n; \tag{4.47}$$

the $2c+1$ eigenstates of C^2 corresponding to a value of l can all be generated from the state with $m = l$ (highest weight state), by succesive application of the operator C^a_-.

For example, in figure 2-1 which shows the eigenstates of four particles, the states at the top of each column are the highest weight states, and the operator C^a_- moves the states one step down in each column. From theorem 1 (page 18) it follows that the color eigenstates with $l = m$ must have a wave function ψ such that when it is antisymmetrized under exchange of its first m and last $n-m$ arguments, equation (2.54), the result is a function χ_m with the following property

$$\left(1 - \sum_{j=1}^{m} \widehat{T}_{j,m+1}\right)\, \chi_m(p_1,\ldots,p_n) = 0, \tag{4.48}$$

This condition implies that χ_m cannot be made more antisymmetric than it already is; to emphasize that property, we write χ_m with a vertical bar separating the groups of arguments under which it is antisymmetric

$$\chi(p_1,\ldots,p_m \mid p_{m+1},\ldots,p_n), \tag{4.49}$$

and equation (4.48) means that the vertical bar cannot be moved to the right.

4.3.1 The fermionic Bethe ansatz

In configuration space, the eigenstates of equation (4.46) take the form

$$|\,\psi\rangle = \int dx_1 \cdots \int dx_n\, \psi(x_1,\ldots,x_n)\, \phi^\dagger_a(x_1)\cdots\phi^\dagger_a(x_m)\phi^\dagger_b(x_{m+1})\cdots\phi^\dagger_b(x_n)\,|\,0\rangle, \tag{4.50}$$

where the wave function ψ has the property

$$A_n(1,\ldots,m)\,A_n(m+1,\ldots,n)\,\psi(x_1,\ldots,x_n) = \chi(x_1,\ldots,x_m \mid x_{m+1},\ldots,x_n); \qquad (4.51)$$

the antisymmetrizers A_n have been defined in chapter 2. Using the configuration-space representation of the Hamiltonian, equation (4.40), and multiplying both sides in equation (4.45) by the vector

$$\langle 0 \mid A_{\alpha_n,p_n}\cdots A_{\alpha_1,p_1} \qquad (\alpha_i = a,b),$$

leads to

$$\left[-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + 2c\sum_{1\le j<l\le n}\delta(x_j-x_l)\right]\Psi(1,\ldots,n) = \lambda\,\Psi(1,\ldots,n), \qquad (4.52)$$

where the complete wave function Ψ has been defined as

$$\Psi(1,\ldots,n) \equiv \langle 0 \mid A_{\alpha_n,p_n}\cdots A_{\alpha_1,p_1} \mid \psi\rangle. \qquad (4.53)$$

After a little algebra, which we will not go through here, Ψ can be written in the form

$$\Psi(1,\cdots,n) = \sum_{\nu}\chi(x_{\nu_1},\ldots,x_{\nu_m} \mid x_{\nu_{m+1}},\ldots,x_{\nu_n})\,\gamma(\alpha_{\nu_1},\ldots,\alpha_{\nu_m} \mid \alpha_{\nu_{m+1}},\ldots,\alpha_{\nu_n}); \qquad (4.54)$$

the functions γ are the color part of the complete wave function. The sum runs only over those permutations ν of S_n which are necessary to generate a basis of the functions with the symmetry of the Young *tableau* $[m, n-m]$.

The eigenvalue problem then reduces to solving the following differential equation

$$\left[-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + 2c\sum_{1\le j<l\le n}\delta(x_j-x_l)\right]\chi(x_1,\ldots,x_m \mid x_{m+1},\ldots,x_n)$$
$$= E\,\chi(x_1,\ldots,x_m \mid x_{m+1},\ldots,x_n), \qquad (4.55)$$

which is the same many-body problem discussed in section 3.2, but for wave functions with the symmetry of a two column *tableau*. Its solution was published independently by Yang [38] and Gaudin [39] in 1967; the wave functions χ are of the Bethe *ansatz* type

$$\chi(x_1,\ldots,x_m \mid x_{m+1},\ldots,x_n) = \sum_{\mu,\nu\in S_n}\xi_{\nu,\mu}\,\theta(x_{\nu_1},\ldots,x_{\nu_n})\,\exp\left(i\sum_{j=1}^{n}\frac{2\pi}{L}K_{\mu_j}x_{\nu_j}\right). \qquad (4.56)$$

As in the case of the field without internal symmetry, there is a set of Bethe momenta $\{K_j\}$ which governs the dynamics of the system. The coefficients a_μ of the bosonic Bethe *ansatz*, equation (4.6), have been replaced by a set of $n!$ dimensional vectors $\vec{\xi}_\mu$.

The $n!$ vectors $\vec{\xi}_\mu$ can be derived from $\vec{\xi}_1$ (1 is the identity permutation). The symmetry of the wave function together with the imposition of periodic boundary conditions and the discontinuities that the derivatives must have, lead to a system of equations which define the allowed Bethe momenta [38, eqs. 20, 22], [41, eqs. 11.58, 11.70]

$$\prod_{\alpha=1}^{n-m} \frac{K_j - \Lambda^\alpha + ic/2}{K_j - \Lambda^\alpha - ic/2} = e^{iLK_j}, \qquad j = 1, \ldots, n \tag{4.57}$$

$$\prod_{j=1}^{n} \frac{K_j - \Lambda^\alpha + ic/2}{K_j - \Lambda^\alpha - ic/2} = -\prod_{\beta=1}^{n-m} \frac{\Lambda^\beta - \Lambda^\alpha + ic}{\Lambda^\beta - \Lambda^\alpha - ic}, \quad \alpha = 1, \ldots, n-m; \tag{4.58}$$

this system of $2n - m$ equations are the so called *Bethe ansatz equations*. In contrast to the case with no internal symmetry, there is an additional set of $n - m$ auxiliary quantities $\{\Lambda^\alpha\}$, but there are also $n - m$ more equations. By taking the logarithm on both sides and using relation (4.18) the equations become

$$K_j = \frac{2\pi}{L} n_j + \sum_{\alpha=1}^{n-m} k_j^\alpha, \qquad j = 1, \ldots, n, \tag{4.59}$$

$$\sum_{j=1}^{n} k_j^\alpha = \frac{2\pi}{L} \lambda^\alpha + \sum_{\beta=1}^{n-m} \omega^{\beta,\alpha}, \qquad \alpha = 1, \ldots, n-m; \tag{4.60}$$

where the numbers n_j and λ^α are integers and the *auxiliary Bethe momenta* k_j^α and $\omega_{\alpha,\beta}$ are defined by

$$k_j^\alpha = \frac{2}{L} \cot^{-1}\left[\frac{2(K_j - \Lambda^\alpha)}{c}\right], \tag{4.61}$$

$$\omega^{\alpha,\beta} = \frac{2}{L} \cot^{-1}\left[\frac{\Lambda^\alpha - \Lambda^\beta}{c}\right], \quad \alpha \neq \beta \tag{4.62}$$

$$\omega^{\alpha,\alpha} = 0. \tag{4.63}$$

For a given set of integers $\{n_1, \ldots, n_n, \lambda_1, \ldots, \lambda_{n-m}\}$ there is a unique solution $\{K_j, \Lambda^\alpha, \lambda^\alpha\}$. The total momentum and energy are

$$Q = \sum_{j=1}^{n} K_j = \sum_{j=1}^{n} n_j + \sum_{\alpha=1}^{n-m} \lambda^\alpha, \tag{4.64}$$

$$E = \sum_{j=1}^{n} K_j^2. \tag{4.65}$$

4.3.2 Momentum-space representation

As we did in the case of bosons (page 47), we will introduce integer dimensionless units in which momenta are measured in units of $2\pi/L$ and energies in units of $2\pi^2/mL^2$. In this system of units the Hamiltonian of equation (4.44) takes the form

$$\widehat{H} = \sum_p p^2 \left(a_p^\dagger a_p + b_p^\dagger b_p\right) + \frac{cL}{2\pi^2} \sum_{p,q,r} a_{p+q}^\dagger b_{r-q}^\dagger b_r a_p; \tag{4.66}$$

its eigenstates are the momentum representation of the fermionic Bethe *ansatz*. They have been derived by Kebukawa [22].

The system that we are studying has Galilean invariance; therefore, it should be possible to write the eigenstates in the simple form of equation (2.38)

$$|\psi\rangle = \sum_{p_1,\cdots,p_n} \delta(p_1+\cdots+p_n-Q)\, \psi(p_1-Q/n,\ldots,p_n-Q/n)\, a_{p_1}^\dagger \cdots a_{p_m}^\dagger b_{p_{m+1}}^\dagger \cdots b_{p_n}^\dagger\, |0\rangle; \tag{4.67}$$

this state has total momentum Q. The wave function $\psi(p_1,\ldots,p_n)$ is a function of the Bethe momenta which are defined by the equations

$$K_j = \frac{m_j}{n} + \sum_{\alpha=1}^{n-m} k_j^\alpha, \qquad j = 1,\ldots,n, \tag{4.68}$$

$$\sum_{j=1}^{n} k_j^\alpha = \lambda^\alpha + \sum_{\beta=1}^{n-m} \omega^{\beta,\alpha}, \qquad \alpha = 1,\ldots,n-m; \tag{4.69}$$

where the *auxiliary momenta* k_j^α and $\omega_{\alpha,\beta}$ are defined as

$$k_j^\alpha = \frac{1}{\pi} \cot^{-1}\left[\frac{4\pi(K_j - \Lambda^\alpha)}{cL}\right], \tag{4.70}$$

$$\omega^{\alpha,\beta} = \frac{1}{\pi} \cot^{-1}\left[\frac{2\pi(\Lambda^\alpha - \Lambda^\beta)}{cL}\right], \qquad \alpha \neq \beta \tag{4.71}$$

$$\omega^{\alpha,\alpha} = 0. \tag{4.72}$$

the quantum numbers n_j and λ^α are integers which must satisfy the following conditions

$$\sum_{j=1}^{n} m_j + n \sum_{\alpha=1}^{n-m} \lambda^\alpha = 0, \tag{4.73}$$

$$m_j \pmod n = Q \pmod n, \qquad j = 1, \ldots, n; \tag{4.74}$$

it is easy to check that

$$\sum_{j=1}^{n} K_j = 0. \tag{4.75}$$

This system of equations are equivalent to the Bethe equations (4.59) and (4.60): if K_j is replaced by $(K_j + Q/n)$ and Λ^α by $(\Lambda^\alpha + Q/n)$ in equations (4.59) through (4.62), and the result written in dimensionless units, we obtain the equations above. Because of the condition imposed on the numbers m_j and λ^α, only $2n - m - 1$ of them are independent; but we also have the quantum number Q.

The eigenstates of only one particle with color b have the following form [22]

$$|\psi\rangle = \frac{1}{\sqrt{(n-1)!}} \sum_{p_1,\ldots,p_n} \psi(p_1 - Q/n, \ldots, p_n - Q/n)\; a^\dagger_{p_1} \cdots a^\dagger_{p_{n-1}} b^\dagger_{p_n} |0\rangle; \tag{4.76}$$

if $|\psi\rangle$ is the highest weight state in a color multiplet, then it has color $c = n/2 - 1$. The wave function ψ is [22]

$$\psi(p_1, \ldots, p_n) = \beta \sum_{\nu \in V} \frac{(-1)^\nu \delta(p_1 + \cdots + p_n)}{(p_1 - K_{\nu_1}) \cdots (p_{n-1} - K_{\nu_{n-1}})}, \tag{4.77}$$

where V is the set of all simple transpositions of the index n with any other indices

$$V = \{T_{1,n}, \ldots, T_{n-1,n}, 1\}, \tag{4.78}$$

and β is a normalization constant.

The bigger the number of particles with color b, the more difficult it is to write the general result of Kebukawa, in a simple form such as equation (4.67). In appendix C we discuss the case of two particles with color b.

Two-fermion system

To close this section we will consider the simple case of only two fermions, one of each color. With $m = 1$ and $n = 2$, equations (4.68) and (4.69) become

$$K_j = \frac{m_j}{2} + k_j, \qquad j = 1, 2 \tag{4.79}$$

$$k_1 + k_2 = \lambda, \tag{4.80}$$

where k_1 and k_2 are given by

$$k_j = \frac{1}{\pi}\cot^{-1}\left[\frac{4\pi(K_j - \Lambda)}{cL}\right], \quad j = 1,2 \tag{4.81}$$

the last two equations can be consistent only if

$$K_1 + K_2 - 2\Lambda = 0; \tag{4.82}$$

but, since the sum of the momenta K_j is zero, Λ must vanish, and the auxiliary momenta become

$$k_1 = \frac{1}{\pi}cot^{-1}\left(\frac{4\pi K_1}{cL}\right) = -k_2. \tag{4.83}$$

This relation leads to

$$m_1 = -m_2 \equiv m \geq 0; \tag{4.84}$$

and there is only one equation left which can written as

$$K = \frac{m}{2} + \frac{1}{\pi}cot^{-1}\left(\frac{4\pi K}{cL}\right), \tag{4.85}$$

where K was defined as

$$K \equiv K_1 = -K_2. \tag{4.86}$$

In view of equation (4.74), the quantum number m must be even if Q is even, and vice versa. The states are

$$|\psi\rangle = \sum_{p_1,p_2} \psi(p_1 - Q/n, p_2 - Q/n)\; a^\dagger_{p_1} b^\dagger_{p_2}\; |0\rangle; \tag{4.87}$$

and the wave function is obtained from equation (4.77)

$$\psi(p_1,p_2) = \frac{\beta\;\delta(p_1+p_2)}{p_1^2 - K^2}. \tag{4.88}$$

Notice that the wave function for the two-boson system, equation (4.38), can be replaced by its completely symmetric component, which is identical to the wave function of the two-fermions. Also, the Bethe equation that defines the momentum K is the same in the two-fermion and two-boson systems; therefore, the energy spectra of the two systems are identical.

4.4 Clustering

The Bethe momenta $\{K_j, \Lambda^\alpha\}$ must be all different. All of the numbers Λ^α are real, while the numbers K_j can be complex only if the interaction is attractive ($c < 0$). The quantum numbers m_j must be all different in the repulsive case. On the other hand, if the interaction is attractive, two quantum numbers m_j can be identical (but no more than two) [39]; for each pair of identical numbers there are two momenta K_j which have a non zero imaginary part.

Gaudin has studied the attractive gas in detail [42, chapter XII]. He has shown that when there is an equal number of fermions of each color, all of the momenta K_j can be complex. There are only $n/2$ quantum numbers which are all repeated twice, and the resulting energy spectrum is very similar to the spectrum of a system of $n/2$ bosons (if the bound states of the bosons are ignored). The spectrum is the same as that of the bosonic Bethe *ansatz* with delta-function interaction of strength $2g$; but the effective interaction among the bosons is repulsive [5]. Each pseudo-boson can be identified as a cluster of two fermions, one of each color.

The energy E is related to the Bethe momenta by equation (4.65); thus

$$\sum_{j=1}^{n} K_j^2 = \text{Real number}; \tag{4.89}$$

this relation, together with (4.75), imply that if one of the Bethe momenta K_j is complex, there must be another momentum K_l which is its complex conjugate. As a consequence, in the case of m clusters there are m distinct complex momenta K_j, and the other m are their complex conjugates. In the next two sections we will study in more detail the systems of one and two clusters.

4.4.1 One-cluster system

The one-cluster system is obtained from the states of two fermions, one of each color — equation (4.87)— if the two momenta K_1 and K_2 are complex. Since the two momenta must

[5] This point is explained in more detail in the next chapter, when the effective hadronic theory is constructed.

be complex conjugate of each other, equation (4.86) implies that K is purely imaginary

$$K = i\alpha, \qquad (\alpha \text{ is real}) \tag{4.90}$$

and equation (4.85) leads to

$$m = 0, \tag{4.91}$$

$$\alpha = -\frac{1}{\pi}\coth^{-1}\left(\frac{4\pi\alpha}{cL}\right). \tag{4.92}$$

The one-cluster state is then

$$|\psi\rangle = \sum_{p_1,p_2} \psi(p_1 - Q/2, p_2 - Q/2)\, a^\dagger_{p_1} b^\dagger_{p_2} |0\rangle, \tag{4.93}$$

where the total momentum Q must be even, in view of equation (4.74), and the wave function, equation (4.88), becomes

$$\psi(p_1,p_2) = \frac{\beta\, \delta(p_1+p_2)}{p_1^2 + \alpha^2}. \tag{4.94}$$

4.4.2 Two-cluster system

We will now construct the states of two clusters. The Bethe *ansatz* for a system of four fermions, two of each color, is (appendix C)

$$|\Omega\rangle = \frac{1}{\sqrt{4}} \sum_{p_1,p_2,p_3,p_4} \Omega(p_1 - Q/4, \ldots, p_4 - Q/4)\, a^\dagger_{p_1} a^\dagger_{p_2} b^\dagger_{p_3} b^\dagger_{p_4} |0\rangle, \tag{4.95}$$

where the wave function is

$$\Omega(p_1,p_2,p_3,p_4) = B \sum_{\nu\in S_4} \frac{(-1)^\nu \delta(p_1+p_2+p_3+p_4)}{(p_3 - K_{\nu_3})(p_4 - K_{\nu_4})} \left[\frac{K_{\nu_1} + \frac{\Lambda^+ + \Lambda^-}{2}}{p_1 - K_{\nu_1}} - \delta_\nu \frac{K_{\nu_1} + K_{\nu_3}}{p_1 + p_3 - K_{\nu_1} - K_{\nu_3}}\right], \tag{4.96}$$

and the number δ_ν is defined as

$$\delta_\nu = \begin{cases} 0, \text{ if } K_{\nu_1} + K_{\nu_3} = 0 \\ 1, \text{ otherwise} \end{cases} \tag{4.97}$$

The state $|\Omega\rangle$ belongs to a color singlet ($c = m - n/2 = 0$). The two-cluster states are obtained if all of the momenta K_j are complex. Two of them must be the complex conjugate of the other two

$$K_1 = K_2^\star, \qquad K_3 = K_4^\star; \tag{4.98}$$

and since their sum must vanish, there is only one independent quantity K from which all four momenta can be derived

$$K_1 = K_2^* = -K_3^* = -K_4 \equiv K. \tag{4.99}$$

As a consequence, the quantum numbers must have the following properties

$$m_1 = m_2 = -m_3 = -m_4 \equiv 2m, \tag{4.100}$$

$$\lambda^1 = \lambda^2 = 0, \tag{4.101}$$

$$\Lambda^1 = -\Lambda^2 \equiv \Lambda. \tag{4.102}$$

The number m can be assumed to be positive; it cannot be zero, because the four numbers m_j cannot be all equal (there are no clusters of four quarks). In principle, m can then be any positive integer divided by two; however, the numbers m_j must satisfy equation (4.74), which in terms of m becomes

$$2m \pmod 4 = -2m \pmod 4 = Q \pmod 4; \tag{4.103}$$

these equations can be valid only if m and $Q/2$ are integers, and they are both either even or odd; hence, the allowed quantum numbers are

$$m = 1, 2, 3, \ldots \tag{4.104}$$

$$Q = \begin{cases} 0, \pm 4, \pm 8, \pm 12, \ldots \text{ if } m \text{ is even} \\ \pm 2, \pm 6, \pm 10, \ldots \text{ if } m \text{ is odd} \end{cases} \tag{4.105}$$

The Bethe equations (4.68) and (4.69) can be written as only two equations in terms of K and Λ

$$K = \frac{m}{2} + k_1 + k_2, \tag{4.106}$$

$$\omega = 2Re(k_1 - k_2), \tag{4.107}$$

where the auxiliary momenta k_1, k_2 and ω are defined as

$$k_j = \frac{1}{\pi} \cot^{-1} \left[\frac{4\pi(K - (-1)^j \Lambda)}{cL} \right], \tag{4.108}$$

$$\omega = \frac{1}{\pi} \cot^{-1} \left(\frac{4\pi\Lambda}{cL} \right). \tag{4.109}$$

cL	m	K	Λ
-2	1	0.4504 + i 0.2516	0.4520
-2	2	0.9747 + i 0.2470	0.9749
-2	4	1.9873 + i 0.2460	1.9874
-10	1	0.3218 + i 0.8209	0.3227
-10	2	0.8845 + i 0.8092	0.8845
-10	4	1.9381 + i 0.8067	1.9381
-20	1	0.2780 + i 1.5924	0.2778
-20	2	0.8263 + i 1.5919	0.8263
-20	4	1.8886 + i 1.5917	1.8886

Table 4.1: Bethe momenta for the two-cluster system.

We have solved these equations numerically, using the Newton-Raphson algorithm. The results for some values of the parameter cL and the quantum number m are shown in table 4-1 and figure 4-1. As cL goes to zero, $Re\ K$ becomes $m/2$ and as $-cL$ is increased it decreases monotonically towards $(m/2 - 1/4)$. The momentum Λ is greater than the real part of K but it is very close to it; in fact in figure 4-1 the two values cannot be distinguished. The imaginary part of K is greater than $-cL/4\pi$ but it approaches that value as $-cL$ increases (figure 4-1).

The total energy [6] is given by equation (4.65), which as a function of K only, becomes

$$E_m = 4(Re\ K)^2 - 4(Im\ K)^2; \tag{4.110}$$

In figure 4-2 we have plotted the energy per quark on the first four levels, for some range of the parameter cL.

To end this chapter, we will prove some properties of the wave function Ω. Before we list

[6] This is the energy in the zero-momentum frame; in general, a factor $Q^2/4$ should be added. Remember that we are measuring energies in dimensionless units; to convert them into natural units, they must be multiplied times $2\pi^2/mL^2$.

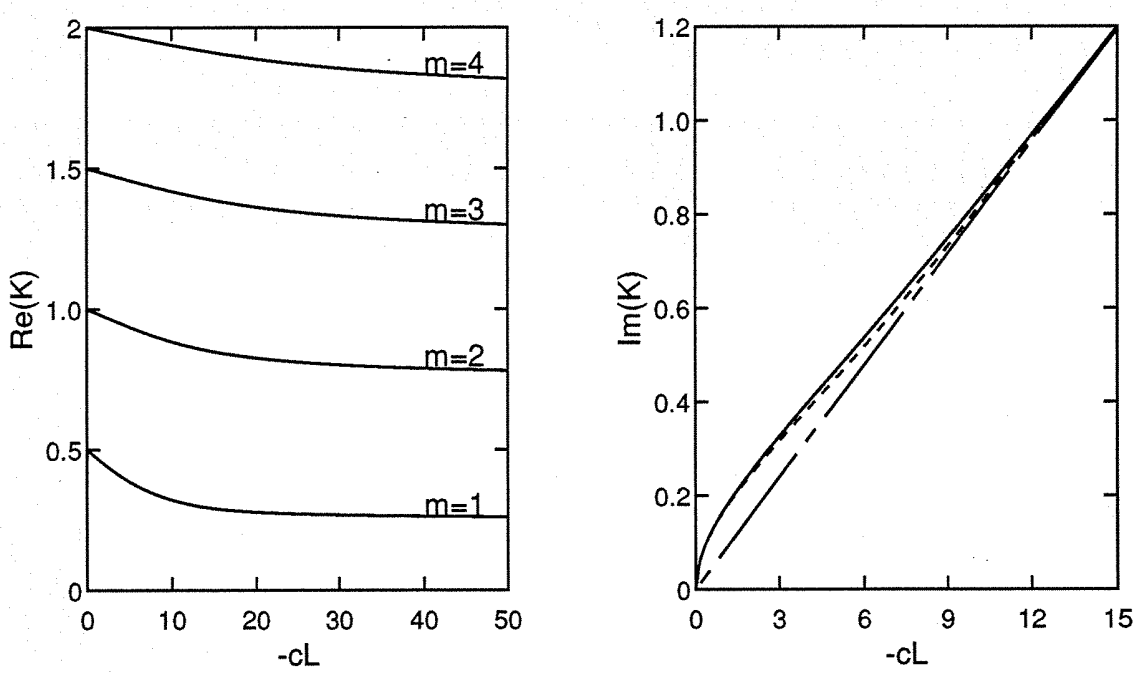


Figure 4-1: Solution to the Bethe equations. Left side: real part of K in the first four levels. Right hand side: imaginary part of K for m=1 (solid) and m=4 (dashed); the broken line is the function $-cL/4\pi$.

the properties, we point out that for a given permutation ν, equation (4.99) implies

$$K_{\mu_j} = (K_{\nu_j})^*, \qquad (j = 1,2,3,4) \tag{4.111}$$

$$K_{\sigma_j} = -K_{\nu_j}, \qquad (j = 1,2,3,4) \tag{4.112}$$

where the permutations μ and σ are defined as

$$\mu = T_{1,2}\, T_{3,4}\, \nu, \tag{4.113}$$

$$\sigma = T_{1,4}\, T_{2,3}\, \nu, \tag{4.114}$$

and $T_{j,l}$ are simple transpositions of the indices j and l (chapter 2).

Proposition 1 $\Omega(p_1,p_2,p_3,p_4)$ *is real.*

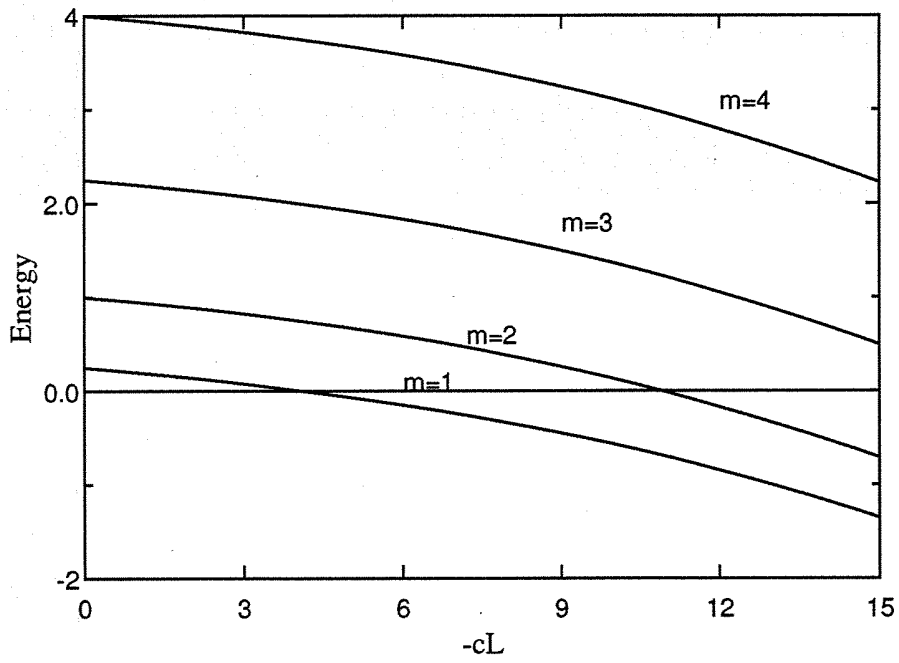


Figure 4-2: Energy spectrum of the two-cluster system.

Proof. Any sum over permutations of order four can be written as

$$\sum_{\nu\in S_4} f_\nu = \sum_{\nu\in V}(f_\nu + f_\mu),$$

where $\mu \equiv T_{1,2}\ T_{3,4}\ \nu$, and V is the set of the twelve permutations such that $\bar{\nu}_1 < \bar{\nu}_2$ ($\bar{\nu}$ stands for the inverse of ν); in simpler terms, V is the set of permutations $(\nu_1,\nu_2,\nu_3,\nu_4)$ in which the digit 1 appears to the left of the digit 2. The wave function Ω has the form

$$\Omega(p_1,p_2,p_3,p_4) = B\ \delta(p_1+p_2+p_3+p_4) \sum_{\nu\in S_4} f_\nu, \tag{4.115}$$

$$f_\nu \equiv \frac{(-1)^\nu}{(p_3 - K_{\nu_3})(p_4 - K_{\nu_4})}\left[\frac{K_{\nu_1}}{p_1 - K_{\nu_1}} - \delta_\nu \frac{K_{\nu_1} + K_{\nu_3}}{p_1+p_3-K_{\nu_1}-K_{\nu_3}}\right]. \tag{4.116}$$

Since the numbers p_j are real, the complex conjugate of f_ν is

$$f^\star_\nu \equiv \frac{(-1)^\nu}{(p_3 - K^\star_{\nu_3})(p_4 - K^\star_{\nu_4})}\left[\frac{K^\star_{\nu_1}}{p_1 - K^\star_{\nu_1}} - \delta_\nu \frac{K^\star_{\nu_1} + K^\star_{\nu_3}}{p_1+p_3-K^\star_{\nu_1}-K^\star_{\nu_3}}\right];$$

and using equation (4.111) together with the facts that $\delta_\nu = \delta_\mu$ and $(-1)^\nu = (-1)^\mu$, it follows that

$$f_\nu^\star = f_\mu,$$

which implies

$$\Omega(p_1,p_2,p_3,p_4) = B\ \delta(p_1+p_2+p_3+p_4)\ 2\ Re\left(\sum_{\nu\in V} f_\nu\right);$$

the normalization constant B can be chosen to be real, which makes Ω real.

Proposition 2 $\Omega(p_1,p_2,p_3,p_4) = \Omega(p_3,p_2,p_1,p_4)$

Proof. If we define

$$\mu \equiv \nu\ T_{1,3},$$

then

$$\mu_1 = \nu_3, \qquad \mu_2 = \nu_2,$$

$$\mu_3 = \nu_1, \qquad \mu_4 = \nu_4,$$

and the sum over the symmetric group can be written in the form

$$\sum_{\nu\in S_4} f_\nu(p_1,p_2,p_3,p_4) = \sum_{\nu\in S_4} f_\mu(p_1,p_2,p_3,p_4). \tag{4.117}$$

If $\delta_\mu = 0$, then $K_{\mu_1} = -K_{\mu_3}$ and

$$\begin{aligned} f_\mu(p_1,p_2,p_3,p_4) &= -\frac{(-1)^\mu\ K_{\mu_3}}{(p_1-K_{\mu_1})(p_3-K_{\mu_3})(p_4-K_{\mu_4})} \\ &= \frac{(-1)^\nu\ K_{\nu_1}}{(p_1-K_{\nu_3})(p_3-K_{\nu_1})(p_4-K_{\nu_4})} \\ &= f_\nu(p_3,p_2,p_1,p_4); \end{aligned}$$

in the second step we used the relation $(-1)^\nu = -(-1)^\mu$.

If $\delta_\mu = 1$, then

$$\begin{aligned} f_\mu(p_1,p_2,p_3,p_4) &= \frac{(-1)^\mu(K_{\mu_1}p_3 - K_{\mu_3}p_1)}{(p_1-K_{\mu_1})(p_3-K_{\mu_3})(p_4-K_{\mu_4})(p_1+p_3-K_{\mu_1}-K_{\mu_3})} \\ &= \frac{(-1)^\nu(K_{\nu_1}p_1 - K_{\nu_3}p_3)}{(p_1-K_{\nu_3})(p_3-K_{\nu_1})(p_4-K_{\nu_4})(p_1+p_3-K_{\nu_1}-K_{\nu_3})} \\ &= f_\nu(p_3,p_2,p_1,p_4); \end{aligned}$$

thus Ω can be written as

$$\begin{aligned}\Omega(p_1,p_2,p_3,p_4) &= B\ \delta(p_1+p_2+p_3+p_4)\sum_{\nu\in S_4} f_\nu(p_3,p_2,p_1,p_4)\\ &= \Omega(p_3,p_2,p_1,p_4).\end{aligned}$$

Proposition 3 $\Omega(p_1,p_2,p_3,p_4)-\Omega(p_2,p_1,p_3,p_4)=\Omega(p_2,p_1,p_4,p_3)-\Omega(p_1,p_2,p_4,p_3)$

Proof.

$$\begin{aligned}&\Omega(p_2,p_1,p_4,p_3)-\Omega(p_1,p_2,p_4,p_3)\\ &= B\ \delta(p_1+p_2+p_3+p_4)\sum_{\nu\in S_4}\frac{(-1)^\nu}{(p_4-K_{\nu_3})(p_3-K_{\nu_4})}\\ &\quad\times\left[\frac{K_{\nu_1}}{p_2-K_{\nu_1}}-\frac{K_{\nu_1}}{p_1-K_{\nu_1}}+\delta_\nu\frac{K_{\nu_1}+K_{\nu_3}}{p_1+p_4-K_{\nu_1}-K_{\nu_3}}-\delta_\nu\frac{K_{\nu_1}+K_{\nu_3}}{p_2+p_4-K_{\nu_1}-K_{\nu_3}}\right],\\ &= B\ \delta(p_1+p_2+p_3+p_4)\sum_{\nu\in S_4}\frac{(-1)^\nu}{(p_3-K_{\nu_3})(p_4-K_{\nu_4})}\\ &\quad\times\left[\frac{K_{\nu_1}}{p_1-K_{\nu_1}}-\frac{K_{\nu_1}}{p_2-K_{\nu_1}}+\delta_\nu\frac{K_{\nu_2}+K_{\nu_4}}{p_1+p_4-K_{\nu_2}-K_{\nu_4}}-\delta_\nu\frac{K_{\nu_2}+K_{\nu_4}}{p_2+p_4-K_{\nu_2}-K_{\nu_3}}\right].\end{aligned}$$

Since the sum of the four Bethe momenta is zero,

$$K_{\nu_2}+K_{\nu_4}=-K_{\nu_1}-K_{\nu_3},$$

and we have

$$\begin{aligned}&\Omega(p_2,p_1,p_4,p_3)-\Omega(p_1,p_2,p_4,p_3)\\ &= B\ \delta(p_1+p_2+p_3+p_4)\sum_{\nu\in S_4}\frac{(-1)^\nu}{(p_3-K_{\nu_3})(p_4-K_{\nu_4})}\\ &\quad\times\left[\frac{K_{\nu_1}}{p_1-K_{\nu_1}}-\frac{K_{\nu_1}}{p_2-K_{\nu_1}}+\delta_\nu\frac{K_{\nu_1}+K_{\nu_3}}{p_2+p_3-K_{\nu_1}-K_{\nu_3}}-\delta_\nu\frac{K_{\nu_1}+K_{\nu_3}}{p_1+p_3-K_{\nu_1}-K_{\nu_3}}\right],\\ &= \Omega(p_1,p_2,p_3,p_4)-\Omega(p_2,p_1,p_3,p_4).\end{aligned}$$

The last two propositions can be used to check that the state $|\,\Omega\rangle$ in fact belongs to a color singlet. The antisymmetrized wave function, equation (2.54), is defined as

$$\chi(p_1,p_2,p_3,p_4)=A_4(1,2)\ A_4(3,4)\ \Omega(p_1,p_2,p_3,p_4); \tag{4.118}$$

using proposition 3, we have

$$\chi(p_1,p_2,p_3,p_4)=\frac{1}{2}\left[\Omega(p_1,p_2,p_3,p_4)-\Omega(p_2,p_1,p_3,p_4)\right]. \tag{4.119}$$

With proposition 2 it is easy to check that

$$A_4(1,2,3)\,\chi = \chi(p_1,p_2,p_3,p_4) - \chi(p_1,p_3,p_2,p_4) - \chi(p_3,p_2,p_1,p_4) = 0, \tag{4.120}$$

which implies that χ cannot be antisymmetrized in the first three arguments. This is the condition imposed on a color singlet (Theorem 1, chapter 2).

We have rewritten the momentum representation of the Bethe *ansatz* in a form simpler than that found in reference [22]. This simplification will prove to be crucial for the computation of observables, in the case of a small system (chapter 6). Hopefully, in the future this approach will help solve the problem of calculating correlations in general.

Chapter 5

A quark model of the nucleus

In chapter 4 we have discussed a many-fermion system which is exactly solvable. We will now develop a quark model of the nucleus which leads to the same Hamiltonian of chapter 4. This work is based on the model for nuclear matter introduced by Koltun *et al* [17,18,19,20]; in their model they studied the ground state of infinite nuclear matter. We will extend the model to the study of dynamical functions of a finite system.

A deuteron-like system is studied, whose eigenstates are given by the two cluster states introduced in chapter 4. From the quark Hamiltonian an effective hadron theory is extracted, in which each cluster is viewed as an elementary particle; the Hamiltonian for the system of clusters (nucleons) is also of the Bethe *ansatz* type. And from the effective theory two approximation schemes are constructed: the impulse approximation and the cluster approximation. The quark model and the two approximations will be used in later chapters to search for experiments which could not be explained with the traditional picture of the nucleus.

5.1 The quark model

We consider the nucleus as a one-dimensional system of quarks. The quarks are of fermionic nature and interact through an attractive delta-function potential. Only the minimum aspects of quark dynamics that may lead to quark effects in the nucleus are retained; thus, the only internal degree of freedom that we will consider is *color*.

As explained in chapter 4, such a system of fermions exhibits the phenomenon of clustering; the maximum number of quarks in each cluster is limited by the number of values that the internal degree of freedom (color) can take. Another remarkable feature of the system is that the clusters behave like a gas of fermions, even in the case of only two colors (clusters of two quarks). As a consequence, the two and three-color cases lead to similar results [1].

In view of the similarity between the two and three-color problems, and in order to keep the calculations as simple as possible, our treatment will be restricted to the two-color case; thus, the nucleons in our deuteron have only two quarks. It remains as a future goal of our research to extend the results of this dissertation to the three color case.

The attractive delta-function interaction binds the quarks into nucleons; and the nucleons have an effective repulsive interaction as we will see later in the chapter. The long range attraction among nucleons which keeps them bound in the nucleus is represented by placing the system inside a finite volume L, of nuclear dimension. The model is solved by means of the Bethe *ansatz*, if periodic boundary conditions are imposed (chapter 4). This choice of boundary conditions leads also to realistic scattering results; i.e. the target is translationally invariant. However, the price we pay for obtaining an exactly solvable model is the loss of the continuity of dynamical functions of momentum: the observables that we will calculate in chapter 6 are all discrete functions of momentum.

The Hamiltonian of the quark system described above is exactly the same that was studied in chapter 4; in momentum representation, it becomes

$$\widehat{H}_q = \sum_p p^2 \left(a_p^\dagger a_p + b_p^\dagger b_p\right) + \frac{cL}{2\pi^2} \sum_{p_1,p_2,r} a_{p_1+r}^\dagger b_{p_2-r}^\dagger b_{p_2} a_{p_1}\,, \tag{5.1}$$

where $c = m_q g$, the constant m_q is the mass of the quarks and g is the strength of the interaction; since the interaction is attractive, the constant g is negative.

5.1.1 Physical parameters and units

There are three physical parameters in the quark model: the nuclear volume L, the mass of the quarks m_q, and the strength of the interaction g. The mathematical solution of the

[1] See for example the discussion of the ground state of nuclear matter given by Tosa [18].

model, investigated in chapter 4, is governed by only one dimensionless parameter

$$cL = m_q g L. \qquad (c < 0) \tag{5.2}$$

In the interpretation of the physical meaning of the mathematical solution, L fixes the momentum scale —the dimensionless momenta of chapter 4 must be multiplied by $2\pi/L$ to convert them into natural units. And both m_q and L determine the energy scale —dimensionless energies are transformed into natural units by multiplying them by $2\pi^2/m_q L^2$.

The parameters m_q and g in our model are not directly measurable; they must be adjusted to obtain agreement with some other empirically known properties of the nucleus. Therefore, it is more convenient to define energy and momentum scales in terms of two constants R and b defined as follows

$$R \equiv -\frac{1}{c}, \tag{5.3}$$

$$b \equiv \frac{c^2}{2m_q}. \tag{5.4}$$

All lengths will be given in units of R, momenta in units of $1/R$ and energies in units of b. We will leave the Hamiltonian, wave functions and Bethe momenta in dimensionless units, as in chapter 4, in order to make the algebra of chapter 6 simpler. The procedure that we will use is the following: first, the value of the nuclear volume, in units of R, is given:L/R. This value is used as the parameter $-cL$ in the equations of chapter 4. The equations are used in dimensionless form. Finally, when we show the results, we transform all momenta into units of $1/R$ by multiplying them by $2\pi R/L$; and energies are given in units of b by multiplying them by $(2\pi R/L)^2$.

We will now explain the physical meaning of R and b. In section 4.4 it was mentioned that a one-cluster system has only one Bethe momentum α which is the solution to equation (4.92)[2]. For a fixed value of c, if the volume L is very large, then α goes to $c/2$, in natural units. Thus $R = -1/c$ is a proper length associated with the cluster (nucleon); it will be shown in chapter 6 that R gives a good measure of the diameter of the nucleon. Then, the parameter L/R measures the ratio of the nuclear volume to the diameter of the nucleon.

[2]Its solution appears in figure 5-2 as the dashed curve.

In a real deuteron, that ratio is of the order of 2.5. However, our model deuteron is one dimensional, and the value of the parameter L/R that gives us a more realistic value of the energy is 5, as we will explain when we discuss the energy spectrum at the end of this section.

According to equation (4.65), the energy of the two-fermion system (in natural units) is

$$E = \frac{2\pi^2}{m_q L^2}(K_1^2 + K_2^2); \tag{5.5}$$

for the isolated nucleon

$$K_1 = -K_2 \approx i\frac{cL}{4\pi}, \tag{5.6}$$

and the binding energy becomes

$$E = -\frac{c^2}{4m_q}; \tag{5.7}$$

therefore, b is twice the binding energy of the nucleon. The mass of the nucleon, M, equals the mass of the two quarks minus the binding energy E

$$M = 2m_q - \frac{b}{2} = \frac{1}{bR^2} - \frac{b}{2}. \tag{5.8}$$

The parameter R is of the order of a fermi, and the mass of the nucleon of the order of 1000 MeV; therefore, the energy and momentum scales that we use are of the order

$$b \approx 40\ MeV, \tag{5.9}$$

$$\frac{1}{R} \approx 200\ MeV. \tag{5.10}$$

5.1.2 Nuclear states in the quark model

Since we are trying to model a deuteron in which each nucleon is a cluster of quarks, we consider only cluster states. The states of the deuteron system are the two-cluster states introduced in section 4.4.2. The total momentum Q has to be even, and $Q/2$ must be odd if the quantum number m is odd and vice versa; therefore, in the center of momentum frame the states can be written as

$$|\Omega\rangle = \frac{1}{\sqrt{4}} \sum_{(r_i+\epsilon)_{i=1\ldots4}} \Omega(r_1, r_2, r_3, r_4)\ a^\dagger_{r_1} a^\dagger_{r_2} b^\dagger_{r_3} b^\dagger_{r_4}\ |0\rangle, \tag{5.11}$$

where ϵ is either 0 or 1/2, depending on whether m is even or odd

$$\epsilon = m \pmod 2/2. \tag{5.12}$$

If $\epsilon = 1/2$, the sum over $r_i + 1/2$ stands for the sum over the sequence of rational numbers $r_i = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$; the wave function is

$$\Omega(r_1, r_2, r_3, r_4) = B \sum_{\nu \in S_4} \frac{(-1)^\nu \delta\,(r_1+r_2+r_3+r_4)}{(r_3 - K_{\nu_3})(r_4 - K_{\nu_4})} \left[\frac{K_{\nu_1}}{r_1 - K_{\nu_1}} - \delta_\nu \frac{K_{\nu_1} + K_{\nu_3}}{r_1+r_3-K_{\nu_1}-K_{\nu_3}} \right], \tag{5.13}$$

where B is a normalization constant. The two-cluster states are completely determined by only one independent momentum K. This Bethe momentum is uniquely defined by a quantum number m and the parameter L, through the equations given in section 4.4.3

$$K = \frac{m}{2} + \frac{1}{\pi} \cot^{-1} \left[\frac{4\pi(K+\Lambda)}{cL} \right] + \frac{1}{\pi} \cot^{-1} \left[\frac{4\pi(K-\Lambda)}{cL} \right], \tag{5.14}$$

$$\cot^{-1} \left(\frac{4\pi\Lambda}{cL} \right) = 2\, Re \left\{ \cot^{-1} \left[\frac{4\pi(K+\Lambda)}{cL} \right] - \cot^{-1} \left[\frac{4\pi(K-\Lambda)}{cL} \right] \right\}. \tag{5.15}$$

Figure 5-1 [3] shows the real part of K in the first four states, and for different values of L; as L goes to zero, $Re(K)$ goes to infinity and as L increases it goes to zero. The imaginary part of K appears in figure 5-2, for the ground state; it is very close to the momentum α of the one-cluster system (chapter 4). It also goes asymptotically to $1/2R$. The imaginary part of K becomes even closer to the momentum of the one-cluster system, for larger values of the quantum number m.

From equation (4.110) we obtain the total energy, which in units of b becomes is

$$E_m = \left(\frac{4\pi R}{L} \right)^2 \left[(Re\ K)^2 - (Im\ K)^2 \right]; \tag{5.16}$$

In figure 5-3 we have plotted the energy in the first four levels, for some range of the parameter L/R. As L goes to infinity, the energy goes asymptotically to -1, which in our system of units corresponds to the internal binding energy of the two clusters. In a real deuteron the nuclear energy is small compared to the internal binding energy of the nucleons; this is true in our model only for $L > 4$, where the ground state energy approaches -1. Then, on the basis of deuteron energetics, we should give a value of the order of 5 to the parameter L/R.

[3] Figures 5-1, and 5-2 are the equivalent of figure 4-1, in the system of units introduced in this chapter.

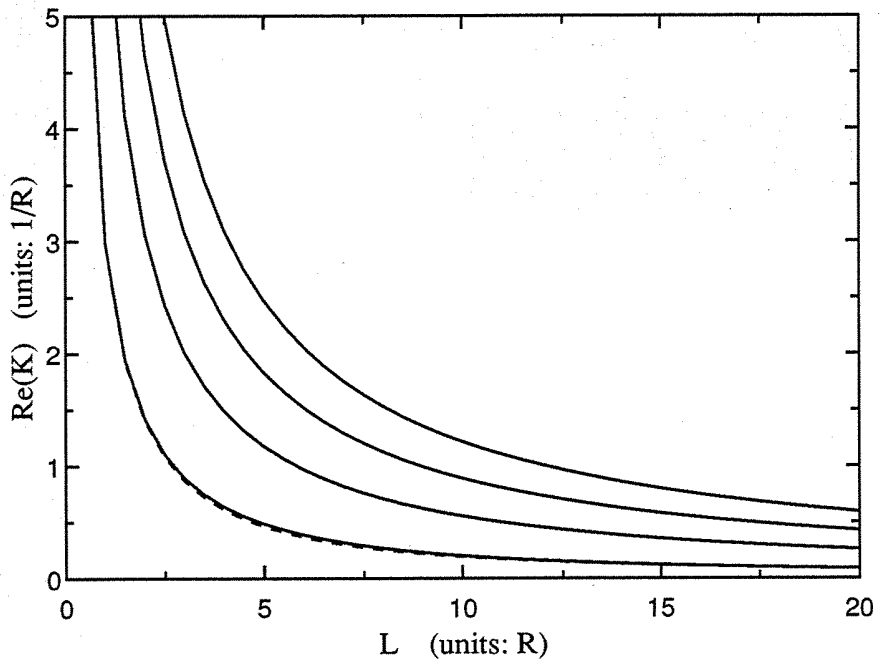


Figure 5-1: Real part of the Bethe momentum of the two-nucleon system, in the first four states. The solid lines are the exact solution, and the dashed lines are obtained with the hadronic theory.

5.2 Hadron picture of the nucleus

The solution to equations (5.14) and (5.15) gives a value of Λ which is always greater than, but very close to $Re\ K$. If we take the limit $\Lambda \to Re\ K^+$, the two equations become

$$k = m + \frac{1}{\pi}\cot^{-1}\left(\frac{2\pi k}{cL}\right), \quad (m = 1, 2, 3, \ldots) \tag{5.17}$$

where $k \equiv 2\ Re(K)$. The imaginary part of K ($\alpha \equiv Im\ K$), can also be calculated up to a good approximation by an equation that involves α and cL only

$$\alpha = -\frac{1}{\pi}\coth^{-1}\left(\frac{4\pi\alpha}{cL}\right). \tag{5.18}$$

In figures 5-1 and 5-2, the results of this approximation (dashed lines) are compared with the exact results. It can be seen that the approximation is very good for the whole range of the parameter L and it becomes even better at large L. Equations (5.14) and (5.15), which

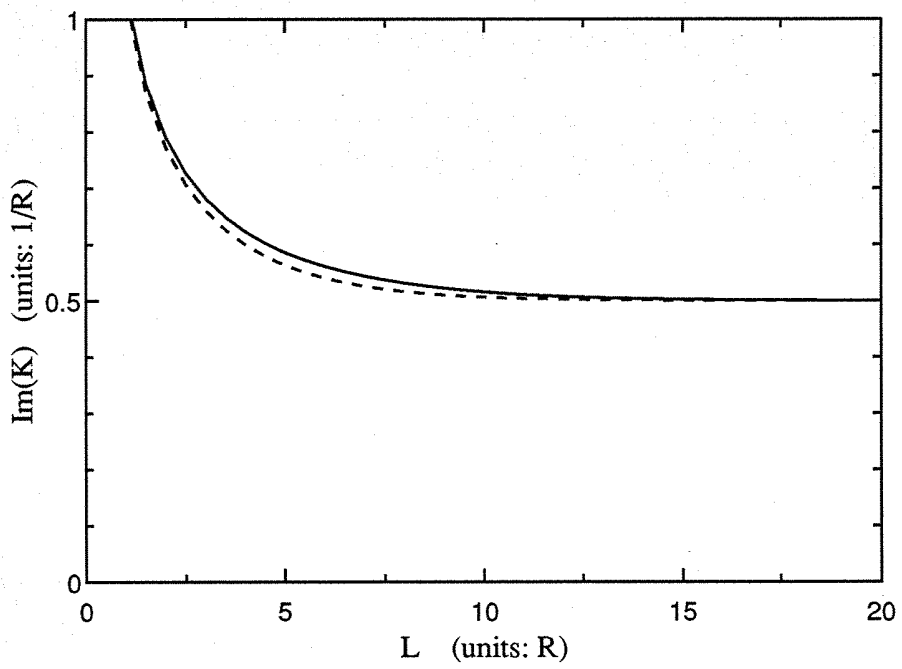


Figure 5-2: Imaginary part of the Bethe momentum of the two-nucleon system, in the ground state (solid line). The dashed line is the Bethe momentum of one cluster. The excited states of the deuteron are all between the solid and the dashed lines, and approach the solution of the cluster as m gets larger.

were coupled, have then been separated into two independent equations that determine the real and imaginary parts of K. Equation (5.18) is the same equation (4.92), of the one-cluster system, and equation (5.17) looks like the equation for the two boson system with delta-function interaction; therefore, this approximation can be used to develop an effective hadron theory.

In the hadron picture, the nucleus is regarded as a system of elementary nucleons, with a spectrum of quantum numbers similar to the momenta of equation (5.17). The imaginary part of K arises from the internal structure of the nucleon, which is treated as a separate problem and its solution requires the study of the quark Hamiltonian but for only one cluster. The fact that nucleons are not truly elementary will be acknowledged in either one of the following two ways: by modifying the nuclear density given by the effective hadronic theory replacing

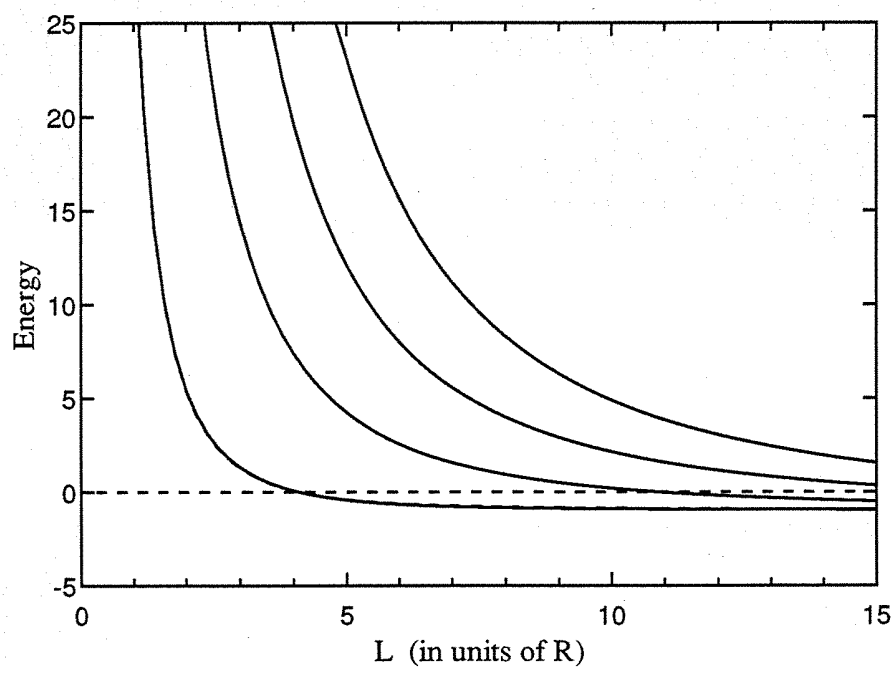


Figure 5-3: Energy spectrum of the two-nucleon system showing the first four states. The energy is given in units of b, ($\approx 40\ MeV$), and the size of the nucleus in units of R ($\approx 1 fm$).

the point charges by the quark distribution of charge of a nucleon (*impulse approximation*), or by modifying the eigenstates of the hadronic Hamiltonian replacing the elementary nucleon operators by the operators that create a nucleon in terms of quarks (*cluster approximation*).

In the next section, we will construct the effective hadronic Hamiltonian which will explain the spectrum given by equation (5.17). In later sections, the eigenstates of the effective Hamiltonian will be used to develop the impulse and cluster approximations to the quark model.

5.2.1 The effective hadron Hamiltonian

Consider a system of bosons, which will represent the *nucleons* in our model, moving inside the one-dimensional finite volume L, with attractive delta-function interaction of strength g

($g < 0$). The Hamiltonian of such a system is (see chapter 4)

$$\widehat{H} = \sum_p p^2 A_p^\dagger A_p + \frac{c_h L}{4\pi^2} \sum_{p_1,p_2,q} A^\dagger_{p_1+q} A^\dagger_{p_2-q} A_{p_2} A_{p_1}, \tag{5.19}$$

where A_p and $A_p^\dagger$ are elementary boson operators with simple commutation relations. The constant c_h is equal to $m_h g$, where m_h is the mass of the nucleon. If we assume that the mass of the nucleons equals the sum of the masses of its two constituent quarks, then $c_h = 2c$, where c is the constant in the quark model. The dimensionless units used here are not the same as in the quark Hamiltonian, because the mass of the quarks m_q is not the same as the mass of the nucleon m_h. To write the Hamiltonian in the same units as the quark Hamiltonian, it must be divided by a factor of two which comes from the ratio nucleon to quark mass. Thus the Hamiltonian for the deuteron-like system, in the hadronic picture becomes

$$\widehat{H}_h = \frac{1}{2}\left[\sum_p p^2 A_p^\dagger A_p + \frac{cL}{2\pi^2} \sum_{p_1,p_2,q} A^\dagger_{p_1+q} A^\dagger_{p_2-q} A_{p_2} A_{p_1}\right]. \tag{5.20}$$

The eigenstates of this Hamiltonian have been given in chapter 4. The overall factor of $1/2$ does not change the states; they are still given by equation (4.37)

$$|\omega_h\rangle = \frac{1}{\sqrt{2}} \sum_{p_1,p_2} \delta(p_1+p_2-Q)\, \omega_h(p_1-Q/2)\, A_{p_1}^\dagger A_{p_2}^\dagger\, |0\rangle; \tag{5.21}$$

and the wave function, equation (4.38), is

$$\omega_h(p) = \frac{\beta_h}{p-k}. \tag{5.22}$$

The equation that defines the Bethe momentum k must be modified, because the parameter $cL/4\pi^2$ that appeared in the Hamiltonian in chapter 4, has now been replaced by $cL/2\pi^2$ in equation (5.20). Equation (4.35) thus becomes

$$k = \frac{l}{2} + \frac{1}{\pi}\cot^{-1}\left(\frac{2\pi k}{cL}\right). \tag{5.23}$$

The overall factor of $1/2$ in the Hamiltonian changes the value of the energy E, making it one half of the value given in chapter 4

$$\widehat{H}\,|\omega_h\rangle = k^2\ |\omega_h\rangle. \tag{5.24}$$

The total momentum Q must be even if l is even and odd if l is odd. The quantum number l can be any positive integer or zero. If it is positive, k is real; and if l were zero, k would be completely imaginary and $|\omega_h\rangle$ would be a bound state of the two bosons. However, comparing equation (5.23) with equation (5.17) which gives the spectrum that we are trying to reproduce, we see that in the effective boson system the quantum number l must be restricted to the sequence $(l = 2, 4, 6, 8, \ldots, 2m)$, that is, any even number except 0.

The hadronic states of the deuteron-like system are then only those states of the two bosons for which the quantum number l is even; and so is the total momentum Q. The exclusion of half of the states from the spectrum of the two-boson system to obtain the hadron spectrum is not at all arbitrary; it has a physical origin which will be explained next. The periodic boundary conditions forbid those momenta of the quarks which are not integers (in the units being used); therefore, if a nucleon is a cluster of two quarks, its momentum must be an even number. The total momentum of the two clusters (Q) then has to be also an even integer. But if the total momentum of the two particles is even, the quantum number l in their Bethe *ansatz* state has to be even. Thus the exclusion of half of the two-boson spectrum results from the periodic boundary conditions together with the fact that the hadrons are not really elementary but made up of two particles.

The two-boson bound state corresponding to $l = 0$ does not appear in the effective hadronic spectrum either; the reason is again the fact that the nucleons are not really bosons but quasibosons; a cluster of two nucleons cannot be allowed because it would represent a cluster of four quarks in a totally symmetric state, which in the case of only two colors constitutes a violation of Pauli's exclusion principle.

Not only should the spectrum be constrained as a consequence of the internal structure of the hadrons, but also the states themselves must be modified. If the clusters are to take only even values of momentum, the states of equation (5.21) must be rewritten in the form

$$|\omega_h\rangle = \frac{1}{\sqrt{2}} \sum_{p_1,p_2} \delta(2p_1+2p_2-Q)\, \omega_h(2p_1-Q/2)\, A^\dagger_{2p_1} A^\dagger_{2p_2} |0\rangle, \qquad (5.25)$$

where the total momentum Q is even; the hadronic wave function ω_h is given by equation (5.22), and k is the solution to equation (5.17), where the quantum number can take the values $m = 1, 2, 3, \ldots$. The hadronic momentum k is such that $k \approx 2ReK$, where K is the

Bethe momentum in the quark model. The energy of the hadronic state $|\omega_h\rangle$ is k^2.

We have thus modified the two-boson Bethe ansatz, equation (5.21), by restricting the sum to only half of the terms. If this were not done, as we calculated observables in the next chapter we would find that all of them would have a domain with twice the number of points than the corresponding observables in the quark model. The choice of only even values for the momenta, has lead to a Kronecker delta function in the state of equation (5.25), which is consistent with equation (4.105).

Even though we started with a system of bosons with attractive interaction, by removing a subset of the energy spectrum we have ended up with the energy spectrum of a system with a repulsive effective interaction. To see why the spectrum corresponds to a repulsive potential, let us notice that in equation (5.17) the solution for a given m is in the interval

$$m - \frac{1}{2} < k < m; \tag{5.26}$$

thus, the m'th energy level is within the interval

$$\left(m - \frac{1}{2}\right)^2 < E_m < m^2. \tag{5.27}$$

But a system of two non-interacting bosons has energy levels

$$E_m^{(0)} = \frac{(m-1)^2}{4}, \quad (m = 1, 2, \ldots) \tag{5.28}$$

which is smaller than E_m for any $m = 1, 2, \ldots$. This means that the *effective interaction* among nucleons is repulsive. We can trace back the physical origin of the repulsive interaction to the fermionic nature of the constituent quarks, which prevents overlapping of the two nucleons thereby excluding the state with $m = 0$. This repulsive interaction prevents the nucleons from binding, which was the reason to introduce a finite volume in our model, in order to confine the nucleons to the nuclear volume.

5.2.2 Internal structure of the nucleon

The internal structure of the nucleon will be calculated by solving the same Hamiltonian of the quark model, but for only a system of two quarks bound into a cluster. The one-cluster

system has been investigated in section 4.4.1; the states have the form

$$|\omega\rangle = \sum_{p_1,p_2} \delta(p_1+p_2-Q)\,\omega(p_1-Q/2)\,a^\dagger_{p_1} b^\dagger_{p_2}\,|0\rangle, \tag{5.29}$$

where the wave function is given by

$$\omega(p) = \frac{\beta}{p^2+\alpha^2}, \tag{5.30}$$

with a normalization constant β; here α is the solution of the equation

$$\alpha = -\frac{1}{\pi}\coth^{-1}\left(\frac{4\pi\alpha}{cL}\right). \tag{5.31}$$

As was discussed earlier, α is such that $\alpha \approx Im(K)$, where K is the Bethe momentum in the quark model. The state of the nucleon $|\omega\rangle$, is an eigenstate of the quark Hamiltonian with energy $-2\alpha^2$

$$H_q\,|\omega\rangle = -2\alpha^2\,|\omega\rangle. \tag{5.32}$$

5.2.3 Impulse approximation

Having developed a hadron theory of the nucleus, and with the internal structure of the nucleon known, an impulse approximation can be introduced. First let us look at the energy of the system; in this approximation it is obtained simply as the sum of the energy of the hadronic states, plus the internal energy of the hadrons. For the deuteron-like system the energy of the hadronic states is given by k^2, and the binding energy of each nucleon is $-2\alpha^2$; therefore, the total energy, in units of b, is

$$E_{IA} = \left(\frac{2\pi R}{L}\right)^2 \left(k^2 - 4\alpha^2\right). \tag{5.33}$$

Since $\alpha \approx Im(K)$ and $k \approx 2Re(K)$, the energy E_{IA} is a good approximation to the quark Hamiltonian energy spectrum, equation (5.16). In fact, in figure 5-3 the results of the impulse approximation cannot be told apart from the quark model spectrum.

Other observables are calculated by folding the quark structure of the hadron with the hadron structure of the nuclear system, as was explained in chapter 3. The impulse approximation simplifies the dynamical problem considerably. The quark Hamiltonian has to be

solved only for the system of quarks inside the hadron; this gives the internal structure of the hadron: binding energy, form factor, momentum distribution, etc. According to this scheme quark degrees of freedom do not play any significant role in nuclear physics; once the intrinsic properties of the nucleons are known from experiments, the dynamical behavior of the system is determined by the effective Hamiltonian.

In spite of the success of the impulse approximation in reproducing the energy spectrum, we notice that the eigenstates in the quark model, equation (5.11), are more complex than the states in the impulse approximation, equation (5.25); therefore, we expect this approximation to fail in the calculation of some other observables. One of the objectives of this dissertation is to look for those observables and to test within which domain the failure becomes significant.

5.2.4 Cluster approximation

The nuclear states are much simpler in the hadron than in the quark picture. We may argue that the apparent simplicity of the state is due to the fact that there is an *internal* part of the wave function, equation (5.29), that is implicitly included in the calculations. In the quark model, the sum over the symmetric group in the wave function and the anti-commutation rules of quark operators ensure that the function has the symmetries required by the identity of quarks. Although the internal wave function of the nucleons is implicitly included in the impulse approximation, quarks belonging to different nucleons are treated as non-identical particles, because they are treated as independent. The impulse approximation does not incorporate exchange of quarks, which one might expect to be a significant source of corrections.

Quark exchange can be included in the hadron picture if the elementary boson operators $A_p^\dagger$ in the hadronic state, equation (5.21), are replaced by quasi-boson operators defined using equation (5.29)

$$\bar{A}_p^\dagger \equiv \sum_{p_1,p_2} \delta(p_1+p_2-p)\, \omega(p_1-p/2)\, a_{p_1}^\dagger b_{p_2}^\dagger; \tag{5.34}$$

the operator $\bar{A}_p^\dagger$ creates a nucleon with momentum p; the algebra of the operators $\bar{A}_p^\dagger$ and $\bar{A}_p$ is no longer given by simple commutation rules, which means that the nucleons are not considered as elementary anymore, but as quasiparticles. Replacing the bosonic operators by

$\bar{A}_p^\dagger$ in equation (5.21) gives

$$|\Omega^h\rangle = \frac{1}{\sqrt{2}} \sum_{p_1,p_2} \sum_{p_3,p_4} \sum_{p_5,p_6} \delta(p_1+p_2-Q)\, \delta(p_3+p_4-p_1)\, \delta(p_5+p_6-p_2) \tag{5.35}$$

$$\times\, \omega_h(p_1 - Q/2)\, \omega(p_3 - p_1/2)\, \omega(p_5 - p_2/2)\, a^\dagger_{p_3} b^\dagger_{p_4} a^\dagger_{p_5} b^\dagger_{p_6}\, |0\rangle; \tag{5.36}$$

evaluation of the sums over p_1 and p_2, and relabeling of the other indices lead to

$$|\Omega^h\rangle = \frac{1}{\sqrt{2}} \sum_{(p_i)_{i=1\ldots4}} \delta(p_1+p_2+p_3+p_4-Q)\, \omega_h(p_1+p_4-Q/2) \tag{5.37}$$

$$\times\, \omega\left(\frac{p_1-p_4}{2}\right)\, \omega\left(\frac{p_2-p_3}{2}\right)\, a^\dagger_{p_1} a^\dagger_{p_2} b^\dagger_{p_3} b^\dagger_{p_4}\, |\, 0\rangle, \tag{5.38}$$

and in the zero-momentum frame the state has the form

$$|\Omega^h\rangle = \frac{1}{\sqrt{4}} \sum_{(r_i+\epsilon)_{i=1\ldots4}} \Omega^h(r_1,r_2,r_3,r_4)\, a^\dagger_{r_1} a^\dagger_{r_2} b^\dagger_{r_3} b^\dagger_{r_4}\, |\, 0\rangle, \tag{5.39}$$

where

$$\epsilon \equiv \frac{m \bmod 2}{2}, \tag{5.40}$$

and the wave function Ω^h is given by

$$\Omega^h(r_1,r_2,r_3,r_4) = -\sqrt{2}\, \delta(r_1+r_2+r_3+r_4)\, \omega_h(r_1+r_3)\, \omega\left(\frac{r_1-r_3}{2}\right)\, \omega\left(\frac{r_2-r_4}{2}\right). \tag{5.41}$$

The states of the nucleus now look more like the states in the quark model. However, the argument of the function ω in equation (5.34) must be an integer. This implies that some of the terms of the sums in the state $|\Omega^h\rangle$ do not give any contribution; namely the terms for which $r_2 - r_4$ and $r_1 - r_3$ are not even integers, which amounts to half of the points in the sums. This is the same phenomenon that was found as we constructed the hadronic states and which had to be imposed in order to get the same number of points in the domain of the observables. However, here the restriction has entered naturally into the picture, because we have aknowledged that the hadronic operators $A_p^\dagger$ are not simple bosonic ones, by replacing them by $\bar{A}_p^\dagger$.

The wave function has the simple product form of equation (1.1). It is not antisymmetric under the exchange of any two arguments; however, since the state is written in terms of quark operators, the result is a state with the correct symmetries imposed by the exclusion

principle. The cluster wave function Ω_h in fact leads to a state which is a color singlet. To check that, first notice that since ω is an even function, the cluster function has the following symmetries

$$\Omega^h(r_1, r_2, r_3, r_4) = \Omega^h(r_3, r_2, r_1, r_4) = \Omega^h(r_1, r_4, r_3, r_2); \tag{5.42}$$

therefore, Ω^h is invariant under the action of the symmetrizers $S_4(1,3)$ and $S_4(2,4)$

$$\Omega^h(r_1, r_2, r_3, r_4) = S_4(1,3)\, S_4(2,4)\, \Omega^h(r_1, r_2, r_3, r_4). \tag{5.43}$$

The antisymmetrized wave function, equation (2.54), can then be written as

$$\chi(r_1, r_2, r_3, r_4) \equiv A_4(1,2)\, A_4(3,4)\, \Omega^h(r_1, r_2, r_3, r_4) = Y(1,2 \mid 3,4)\, \Omega^h(r_1, r_2, r_3, r_4), \tag{5.44}$$

where $Y(1,2 \mid 3,4)$ is the Young operator

$$Y(1,2 \mid 3,4) \equiv A_4(1,2)\, A_4(3,4)\, S_4(1,3)\, S_4(2,4); \tag{5.45}$$

therefore, the function χ has the form

$$\chi(r_1, r_2 \mid r_3, r_4), \tag{5.46}$$

which is the form required for a color singlet (theorem 1, chapter 2).

5.3 Summary

In the quark model the states of the deuteron are known; they have been written in momentum space, which will simplify the evaluation of the observables of interest in scattering theory. The energy spectrum is well reproduced by an effective hadron theory in which the deuteron is regarded as a system of two elementary nucleons, with an effective repulsive interaction. The effective Hamiltonian is also of the Bethe *ansatz* type, but only some of its eigenstates are allowed due to the underlying substructure of nucleons. The binding energy of the nucleons is obtained by solving the quark Hamiltonian for one nucleon only.

An impulse approximation has been introduced, assuming that the only role of quark dynamics in nuclear physics is fixing the internal structure of the nucleon; this substructure is assumed to be an intrinsic property of nucleons which is not affected by the presence of

other nucleons. In the cluster approximation the states of the deuteron in the effective theory are modified by replacing the elementary nucleon operators by the operator that creates a cluster of quarks. This approximation is expected to be an improvement to the impulse approximation because it introduces terms due to exchange of quarks among nucleons; however, there is another important source of error in both approximations: the exact states in the quark model cannot be factored into cluster-cluster and internal parts. In chapter 7 we will see that the effect of the wave function not being separable is more important that the effect of quark antisymmetrization, and thus the impulse approximation is as good as the cluster approximation (it actually gives better results, as we will see in chapter 7).

Chapter 6

Computation of observables

The states of the deuteron, in the quark model and in the two approximations are now used to calculate dynamical functions. These are new calculations; Bethe *ansatz* models have been used in the past to calculate dynamical functions of the nucleus [43,44], but the results found in the literature are for the bosonic Bethe *ansatz*, and only in the case of elastic scattering. The major difficulty in doing such calculations in configuration space, arises from the rapidly increasing number of integrals that have to be evaluated (increasing as $n!^2$). In the simple case of only two clusters, those calculations would be cumbersome, but could still be reasonably done; instead, we have decided to use momentum representation because in that case the space does not have to be separated into 4! regions; in the end, we are also left with $(4!)^2$ terms, but they can be written in a compact and elegant algebraic form that makes it easy to develop an algorithm to obtain the results. We expect that the procedure we have used would be easier to generalize than the configuration space formulation.

In chapters 2 and 3 some dynamical functions were introduced, which can be measured in scattering experiments and could be used to look for quark effects in the nuclear target. Our goal in this chapter is to derive analytic expressions for those functions, both in the quark model and in the approximations based on the effective hadron theory. The observables of interest are: the quark-momentum distribution, the quark-correlation functions, the elastic form factor, and the Coulomb sum rule. The analytic expressions obtained in this chapter will be used in chapter 7 as a tool to look for quark effects in the nucleus, other than those

already present in the nucleon.

The impulse approximation will be studied first, because of the simplicity of its calculations, which are a good introduction to the more general approaches of the cluster approximation and quark model. The cluster approximation constitutes also a more rigorous way of checking the results of the impulse approximation, which were obtained with some physical intuition (section 3.3); the results from the impulse approximation must agree with the direct terms in the cluster approximation.

In both the cluster approximation and the quark model, the state of the system in terms of quarks is known and the calculations are done in a similar way, using the same diagrams for each observable. What is different is the analytic expressions of the diagrams obtained with the quark model of the cluster approximation.

6.1 The impulse approximation

In the impulse approximation, the calculation of observables is done at two independent levels. At the nuclear level, the effective hadron theory is used to calculate nucleon momentum distribution, n_h, elastic form factor, f_h and nucleon-nucleon correlation function, c_h. The quark Hamiltonian is used only to study the system of quarks that form a nucleon. The quark state of the nucleon is used to obtain its elastic form factor, f, quark-momentum distribution, n, and quark-quark correlation function, d. The two sets of results are then combined to calculate the quark functions of the nucleus, as explained in chapter 3.

6.1.1 Quark structure of the nucleon

The intrinsic quark functions of the nucleon can be calculated using the quark state of the nucleon, equation (5.29)

$$|\omega\rangle = \sum_{p_1,p_2} \delta(p_1+p_2-Q)\,\omega(p_1-Q/2)\,a^\dagger_{p_1} b^\dagger_{p_2}\,|0\rangle; \tag{6.1}$$

the wave function $\omega(p)$ is an even function and is given by equation (5.30)

$$\omega(p) = \frac{\beta}{p^2+\alpha^2}. \tag{6.2}$$

The quark observables of the nucleon are derived from the general results found in sections 2.4 and 2.5, for systems of fermions with two colors; the number of particles n, and the number of those with color a, m, are 2 and 1 respectively. Both functions $\psi(p_1,p_2)$ and $\chi(p_1,p_2)$ should replaced by

$$\delta(p_1+p_2-Q)\,\omega(p_1-Q/2).$$

We first have to normalize the state; from equation (2.84)

$$\begin{aligned}\langle\omega\,|\,\omega\rangle &= \sum_{p_1,p_2}\delta(p_1+p_2-Q)\,\omega^2(p_1-Q/2)\\ &= \sum_{p}\omega^2(p);\end{aligned}\tag{6.3}$$

in the last step we used the fact that Q must be an even number (section 4.4.1). With the form of the wave function, equation (6.2), we find that in order to normalize the state, β must be chosen to be

$$\beta=\left[\sum_p (p^2+\alpha^2)^{-2}\right]^{-1/2}=[\xi_4(\pm i\alpha)]^{-1/2}\,,\tag{6.4}$$

where the function ξ_4 is defined in appendix A, and its analytic form is given there.

The quark momentum distribution $n(p)$ is obtained from equation (2.83)

$$\begin{aligned}n(p) &= \frac{1}{2}\sum_{p_1}\delta(p_1+p-Q)\left[\omega^2(p-Q/2)+\omega^2(p_1-Q/2)\right]\\ &= \omega^2(p);\end{aligned}\tag{6.5}$$

the even nature of the function ω and the number Q were used. With the functional form of ω, we have

$$n(p)=\left[\frac{\beta}{p^2+\alpha^2}\right]^2,\qquad\qquad (p \text{ integer}).\tag{6.6}$$

Likewise, the elastic form factor is obtained from equation (2.88)

$$\begin{aligned}f(q) &= \frac{1}{2}\sum_{p_1,p_2}\delta(p_1+p_2-Q)\,\omega(p_1-Q/2)\,[\omega(p_1-Q/2+q/2)+\omega(p_1-Q/2-q/2)]\\ &= \sum_p \omega(p)\,\omega(p+q/2);\end{aligned}\tag{6.7}$$

the momentum transfer q has to be an even number because the argument of ω must be an integer. Using the algebraic form of ω, the sum over p becomes a function ξ_4 (appendix A)

$$f(q)=\beta^2\,\xi_4(q/2\pm i\alpha,\pm i\alpha).\tag{6.8}$$

If $q = 0$, it is clear from equation (6.4) that $f(0) = 1$; if $q \neq 0$, using proposition 19 from appendix A, and equation (5.18) we obtain the result

$$f(q) = -\frac{32\pi^2\beta^2/cL}{q^2 + (4\alpha)^2}, \qquad (q = \pm 2, \pm 4, \pm 6, \ldots). \tag{6.9}$$

The two quarks inside the nucleon have different color; hence, the quark-quark correlation function of the nucleon, $d(q)$ is derived from the expression for the different-color correlation function, equation (2.98)

$$\begin{aligned} d(q) &= \delta(p_1+p_2-Q)\,\omega(p_1-Q/2)\,\omega(p_1-Q/2+q), \\ &= \sum_p \omega(p)\,\omega(p+q); \end{aligned} \tag{6.10}$$

which is identical to $f(2q)$, equation (6.7). This equality is a consequence of having only two quarks inside the nucleon. In any system of two particles with the same mass, the probability of finding the two particles at a distance x from each other ($P_2(x)$), equals the probability of finding one particle at a distance $x/2$ from the center of mass ($\rho(x/2)$). Since $d(q)$ and $f(q)$ are the Fourier transforms of $P_2(x)$ and $\rho(x)$ respectively, it follows that $d(q) = f(2q)$.

Using the result for $f(q)$, we can write the quark-quark correlation function of the nucleon as

$$d(q) = -\frac{8\pi^2\beta^2/cL}{q^2 + 4\alpha^2}, \qquad (q = \pm 1, \pm 2, \pm 3, \ldots). \tag{6.11}$$

The *isolated* nucleon is obtained if we take the limit $L \to \infty$. In that limit, as we saw in the section on parameters and units in chapter 5, the solution of equation (5.18) is

$$\alpha = \frac{cL}{4\pi}. \tag{6.12}$$

The momentum transfer $2\pi q/L$ becomes continuous and the elastic form factor, equation (6.7), becomes

$$f(q) = \frac{4c^2}{q^2 + 4c^2}; \qquad (q \text{ Real}) \tag{6.13}$$

the Fourier transform of this function gives the density of the nucleon:

$$\rho(x) = -ce^{2c\,|x|}; \quad (c < 0) \tag{6.14}$$

the integral of the density from $-\infty$ to ∞ has been normalized to 1. The integral from $1/2c$ to $-1/2c$ is approximately 0.63; therefore, the parameter $R \equiv -1/c$ defined in chapter 5, gives an estimate of the diameter of the isolated nucleon.

6.1.2 Nuclear structure in the effective hadron theory

In the effective hadron theory the states of the deuteron are given by equation (5.25)

$$|\omega_h\rangle = \frac{1}{\sqrt{2}} \sum_{p_1,p_2} \delta(2p_1+2p_2-Q)\, \omega_h(p_1-Q/2)\, A^\dagger_{2p_1} A^\dagger_{2p_2} |0\rangle; \tag{6.15}$$

the total momentum Q is even. The operators $A^\dagger$ are elementary boson operators. The wave function $\omega_h(p)$ has been given in chapter 5, equation (5.22); in order to simplify the algebra, we will use only the even part of $\omega_h(p)$, which leads to the same state $|\omega_h\rangle$; thus, we have

$$\omega_h(p) = \frac{\beta_h}{p^2 - k^2}, \tag{6.16}$$

where β_h is an arbitrary constant. The value of the constant is chosen such that the norm of the state is one

$$\begin{aligned} \langle\omega_h|\omega_h\rangle &= \sum_{p_1,p_2,p_3,p_4} \delta(2p_1+2p_2-Q)\, \delta(2p_3+2p_4-Q) \\ &\quad \times \omega_h(2p_2-Q/2)\, \omega_h(2p_3-Q/2)\, \langle 0|\, A_{2p_1} A_{2p_2} A^\dagger_{2p_3} A^\dagger_{2p_4}\, |0\rangle, \\ &= \sum_p \omega_h^2(2p-Q/2). \end{aligned} \tag{6.17}$$

As we mentioned in chapter 5, $Q/2$ is an integer; and for a given quantum number m the allowed values of $Q/2$ are those such that

$$\frac{Q}{2} \pmod 2 = m \pmod 2; \tag{6.18}$$

if the rational number ϵ is defined as in equation (5.12)

$$\epsilon \equiv \frac{m \pmod 2}{2}, \tag{6.19}$$

the norm becomes

$$\langle\omega_h|\omega_h\rangle = \sum_p \omega_h^2(2p-2\epsilon), \tag{6.20}$$

and from equation (6.16) the normalization constant has the form

$$\beta_h = 4\,[\xi_4(\epsilon \pm K)]^{-1/2}. \tag{6.21}$$

The number ϵ vanishes if m is even, and $\epsilon = 1/2$ otherwise; the real momentum K has been defined as

$$K \equiv \frac{k}{2}. \tag{6.22}$$

To calculate the hadron functions of the deuteron, we will use the definitions in chapter 2, dropping the color indices α in the operators A and $A^\dagger$. From equation (2.36), the nucleon-momentum distribution is

$$\begin{aligned} n_h(p) &= \frac{1}{2}\langle \omega_h |\, A_p^\dagger A_p \,| \omega_h \rangle \\ &= \frac{1}{4} \sum_{p_1,p_2,p_3,p_4} \delta(2p_1+2p_2-Q)\, \delta(2p_3+2p_4-Q)\, \omega_h(2p_2-Q/2), \\ &\quad \times \omega_h(2p_3-Q/2)\, \langle 0 |\, A_{2p_1} A_{2p_2} A_p^\dagger A_p A_{2p_3}^\dagger A_{2p_4}^\dagger \,| 0 \rangle. \end{aligned} \tag{6.23}$$

The argument p must then be even; it is easy to simplify the expression above, leading to the result

$$n_h(p) = \omega_h^2(p-Q/2), \qquad (p \text{ even}). \tag{6.24}$$

The elastic form factor is calculated as a matrix element of the density operator $\widehat{\rho}_q$, equation (2.28). The final state should be the same initial one, but with an overall increase of q in the total momentum. However, since the total momentum of $|\, \omega_h \rangle$ must satisfy the relation

$$Q \pmod 4 = 2\epsilon, \tag{6.25}$$

the momentum transfer q has to be a multiple of 4. From equation (2.28) we have

$$\begin{aligned} f_h(q) &= \frac{1}{2}\langle \omega_h(Q+p) |\, \sum_{p_1} A_{p_1+p}^\dagger A_{p_1} \,| \omega_h(Q) \rangle \\ &= \frac{1}{4} \sum_{p_1,p_2,p_3,p_4} \delta(2p_1+2p_2-Q-p)\, \delta(2p_3+2p_4-Q)\, \omega_h(2p_2-Q/2-p)\, \omega_h(2p_3-Q/2), \\ &= \times \left[\langle 0 |\, A_{2p_1} A_{2p_2} A_{2p_3}^\dagger A_{2p_4+p}^\dagger \,| 0 \rangle + \langle 0 |\, A_{2p_1} A_{2p_2} A_{2p_3+p}^\dagger A_{2p_4+p}^\dagger \,| 0 \rangle \right], \end{aligned} \tag{6.26}$$

where $p = q/4$. After some algebra, f_h can be written as

$$f_h(q) = \sum_p \omega_h(2p-2\epsilon)\, \omega_h(2p+q/2-2\epsilon); \tag{6.27}$$

using the functional form of ω_h, f_h becomes a function ξ_4

$$f_h(q) = \left(\frac{\beta_h}{4}\right)^2 \xi_4(q/4+\epsilon\pm K, \epsilon\pm K). \tag{6.28}$$

If $q = 0$, according to the definition of β_h, $f_h(0) = 1$. At momentum transfer different from zero, proposition 19 leads to the following result

$$f_h(q) = \left(\frac{\beta_h}{4}\right)^2 \frac{32\xi_1(K+\epsilon)/K}{q^2-(4k)^2}, \qquad (q = \pm 4, \pm 8, \pm 12, \ldots). \tag{6.29}$$

The nucleon-nucleon correlation function is defined in terms of the two-particle correlation operator, equation (2.34)

$$c_h(q) = \frac{1}{2}\langle\omega_h| \sum_{p,r} A^\dagger_{p-q} A^\dagger_{r+q} A_r A_p \,|\omega_h\rangle. \tag{6.30}$$

Instead of going through some algebra, we use the result discussed in the previous section for two-particle systems. Since the deuteron is made up of only two nucleons, the nucleon-nucleon correlation function $c_h(q)$ must be equal to the elastic form factor f_h, evaluated at $2q$. Thus, the result is

$$c_h(q) = \left(\frac{\beta_h}{4}\right)^2 \frac{8\xi(K+\epsilon)/K}{q^2-(2k)^2}, \qquad (q = \pm 2, \pm 4, \pm 6, \ldots). \tag{6.31}$$

6.1.3 Quark structure of the deuteron in the impulse approximation

Now that we know the structure of the deuteron in terms of nucleons, and the quark structure of a nucleon, we can calculate the quark functions of the deuteron following the approach outlined in section 3.3. The results from that section will be valid here if the integrals over momenta are replaced by discrete sums, and three-dimensional vectors are made one-dimensional.

From equation (3.38) we have

$$N_{IA}(p) = \sum_q n(p-q/2)\, n_h(q). \tag{6.32}$$

The function n_h is different from zero only if its argument is an even number; therefore, the sum over q can be restricted to even values only. Using the distributions $n(p)$ and $n_h(p)$,

equations (6.6) and (6.24), we obtain

$$N_{IA}(p) = \sum_q \omega^2(q)\, \omega_h^2(2p-2q-Q/2), \tag{6.33}$$

where Q is the total momentum of the nucleus. The momentum transfer in the zero-momentum frame is given by $r \equiv (p - Q/4)$; since Q is constrained by equation (6.25), r is either an integer, or an integer plus 1/2, depending on the value of ϵ; therefore, in the zero-momentum frame the quark-momentum density is

$$N_{IA}(r) = \sum_q \omega^2(q)\, \omega_h^2(2r - 2q), \tag{6.34}$$

where the domain of N_{IA} is

$$r = p - \epsilon, \qquad (p \text{ integer}). \tag{6.35}$$

The quark momentum distribution in the impulse approximation is then the convolution of the distribution of quarks inside the hadron and the distribution of hadrons inside the nucleus. This simple result was derived without any knowledge of the state of the nucleus in terms of quarks. Using the algebraic form of the functions ω and ω_h, equations (6.2) and (6.16), the quark momentum distribution becomes

$$N_{IA}(r) = \beta^2\, \beta_h^2 \sum_p (p^2 + \alpha^2)^{-2} \left[(2r-2p)^2 - k^2\right]^{-2}; \tag{6.36}$$

if a factor of 2 is taken out of the last parenthesis, the sum takes the form of a function ξ_8 (appendix A)

$$N_{IA}(r) = \beta^2 \left(\frac{\beta_h}{4}\right)^2 \xi_8(\pm i\alpha, r \pm K); \tag{6.37}$$

with $K \equiv k/2$. It can be verified that the quark momentum distribution is normalized to one

$$\begin{aligned} \sum_{r+\epsilon} N_{IA}(r) &= \beta^2 \left(\frac{\beta_h}{4}\right)^2 \sum_{p,q} (p^2 + \alpha^2)^{-2} \left[(q-\epsilon-p)^2 - K^2\right]^{-2} \\ &= \beta^2 \left(\frac{\beta_h}{4}\right)^2 \xi_4(\pm i\alpha)\, \xi_4(\epsilon \pm K) \qquad (6.38) \\ &= 1. \qquad (6.39) \end{aligned}$$

According to equation (3.40), the elastic form factor in the impulse approximation is simply the product

$$F_{IA}(q) = f(q)\, f_h(q); \tag{6.40}$$

since the domain of f_h includes only multiples of 4, the elastic form factor F_{IA} is defined only for multiples of 4 also. Using the form factors f and f_h calculated earlier, equation (6.40) becomes

$$F_{IA}(q) = \left(\frac{\beta\beta_h}{4}\right)^2 \frac{2^{10}\pi^2\,\xi_1(K+\epsilon)}{cLK} \frac{1}{(q^2+16\alpha^2)(q^2-16k^2)}, \quad (q = \pm 4, \pm 8, \pm 12, \ldots). \tag{6.41}$$

In the impulse approximation the same-color correlation function is calculated as follows. Two quarks of the same color inside the nucleus can only come from different nucleons; therefore, the probability of finding two quarks of the same color at a distance x from each other is given by equation (3.43)

$$P_c(x) = \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} dy\; dz\; \rho(y)\; \rho(z)\; p_h(x+y-z), \tag{6.42}$$

where $\rho(y)$ is the quark density of the nucleon, and $p_h(x)$ is the probability of finding the two nucleons in the deuteron at a distance x from each other. Both functions are periodic, with period L. The same–color correlation function is simply the Fourier transform of P_c, which has been calculated in section 3.3.3, and the result is

$$C_{IA}(q) = f^2(q)\, c_h(q), \qquad (q = 0, \pm 2, \pm 4, \ldots). \tag{6.43}$$

The different-color quark correlation function is the Fourier transform of the probability of finding two quarks, of different color, at a distance x of each other. If two quarks of different color are randomly picked, there are equal chances that they come either from the same nucleon or from different ones; then the different-color correlation function is obtained from (3.49), giving an equal weight to both terms in the sum

$$D_{IA}(q) = \frac{1}{2}\left[d(q) + f^2(q)\, c_h(q)\right]; \tag{6.44}$$

here d is the quark-quark correlation function of the nucleon, and f its elastic form factor; c_h is the nucleon-nucleon correlation function of the deuteron.

The Coulomb sum rule was calculated in chapter 3, using also the closure approximation. The result given there, equation (3.57), was

$$\Re_{IA}(q) = \frac{f^2(q)}{2}\left[1 + c_h(q)\right] - f^2(q)\, f_h^2(q) \qquad (q = 0, \pm 2, \pm 4, \ldots). \tag{6.45}$$

At zero momentum transfer, c_h, f_h and f are normalized to 1, and $\Re_{IA}$ becomes zero. At large momentum transfer, the form factor of the nucleon, goes to zero and so does $\Re_{IA}$.

6.2 The cluster approximation

In the cluster approximation, the quark functions that we study can all be calculated as matrix elements or expectation values of the density operators defined in chapter 2. The results will be separated into a direct term plus an exchange contribution which arises from the antisymmetrization under exchange of quarks among nucleons.

The computation of observables can be made in an elegant way by introducing *cluster diagrams* [8,9]; diagram rules will be derived in the next section and two standard diagrams will be calculated. Cluster diagrams are not a perturbative approach; there is a finite number of them and they can be calculated analytically.

6.2.1 Cluster diagrams

The states of the duteron, in the cluster approximation were obtained in page 78; it is convenient to rewrite them in the form

$$|\Omega^h\rangle = \frac{1}{4} \sum_{(r_i+\epsilon)_{i=1,2,3,4}} \Omega^h(r_1,r_2,r_3,r_4)\, a^\dagger_{r_1} b^\dagger_{r_2} a^\dagger_{r_3} b^\dagger_{r_4} \,|0\rangle, \tag{6.46}$$

with the wave function

$$\Omega^h(r_1,r_2,r_3,r_4) = \sqrt{2}\,\delta(r_1+r_2+r_3+r_4)\,\omega_h(r_1+r_2)\,\omega\left(\frac{r_1-r_2}{2}\right)\,\omega\left(\frac{r_3-r_4}{2}\right); \tag{6.47}$$

the numbers r_i are integers, if the quantum number m is even ($\epsilon = 0$), and integers plus $1/2$ if m is odd ($\epsilon = 1/2$). The functions ω and ω_h are given by equations (6.2) and (6.16); since they are both even functions, Ω^h has the following properties

$$\Omega^h(r_1,r_2,r_3,r_4) = \Omega^h(r_2,r_1,r_3,r_4), \tag{6.48}$$

$$\Omega^h(r_1,r_2,r_3,r_4) = \Omega^h(r_1,r_2,r_4,r_3), \tag{6.49}$$

$$\Omega^h(r_1,r_2,r_3,r_4) = \Omega^h(r_3,r_4,r_1,r_2), \tag{6.50}$$

$$\Omega^h(r_1,r_2,r_3,r_4) = \Omega^h(-r_1,-r_2,-r_3,-r_4). \tag{6.51}$$

In chapter 2 we derived general expressions for the elastic form factor and correlation functions of a system of particles of two colors. In the case of the deuteron, the results are

all sums of terms of the form

$$\mathcal{D}^h(a_1,a_2,a_3,a_4) = \sum_{(r_i+\epsilon)_{i=1,2,3,4}} \Omega^h(r_1,r_2,r_3,r_4)\,\chi^h(r_1-a_1,r_2-a_2,r_3-a_3,r_4-a_4), \quad (6.52)$$

where χ^h is the wave function antisymmetrized under exchange of quarks of the same color

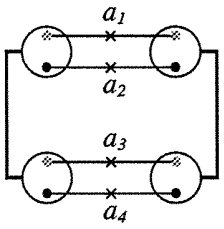
$$\begin{aligned}\chi^h(r_1,r_2,r_3,r_4) &= A_4(1,3)\,A(2,4)\,\Omega^h(r_1,r_2,r_3,r_4)\\ &= \frac{1}{4}\left[\Omega^h(r_1,r_2,r_3,r_4)+\Omega^h(r_3,r_4,r_1,r_2)-\Omega^h(r_3,r_2,r_1,r_4)-\Omega^h(r_1,r_4,r_3,r_2)\right] \quad (6.53)\end{aligned}$$

From the symmetry property (6.50), the sum inside the square brackets reduces to only two terms and $\mathcal{D}^h$ can be written in the form

$$\mathcal{D}^h(a_1,a_2,a_3,a_4) = \frac{1}{2}\left[\mathcal{D}^h_d(a_1,a_2,a_3,a_4)+\mathcal{D}^h_e(a_1,a_2,a_3,a_4)\right], \quad (6.54)$$

where the *direct diagram* $\mathcal{D}^h_d$ is defined as

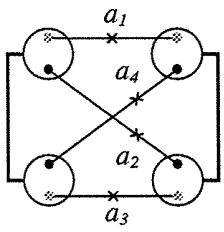
$$\mathcal{D}^h_d(a_1,a_2,a_3,a_4) = \quad (6.55)$$



$$= \sum_{(r_i+\epsilon)_{i=1,2,3,4}} \Omega^h(r_1,r_2,r_3,r_4)\,\Omega^h(r_1-a_1,r_2-a_2,r_3-a_3,r_4-a_4);$$

the *exchange diagram* is [1]

$$\mathcal{D}^h_e(a_1,a_2,a_3,a_4) = \quad (6.56)$$



$$= -\sum_{(r_i+\epsilon)_{i=1,2,3,4}} \Omega^h(r_1,r_2,r_3,r_4)\,\Omega^h(r_1-a_1,r_3-a_4,r_2-a_2,r_4-a_3).$$

The relation between each term and its corresponding diagram should be clear. The four small circles (*quarks*) on the left, correspond to the arguments of the first function Ω^h in order

[1] The symmetry property (6.49) has been used to make the diagram look simpler.

from top to bottom; the quarks on the right represent the arguments of the second function Ω^h. Inside each big circle (*nucleon*) there is one quark of each color; and every quark on the left is linked to one, and only one quark of the same color, on the right. Each segment connecting two quarks is assigned one of the indices a_i in the following order: a_1 goes with the first grey quark on the left, a_2 with the first black quark on the left, a_3 with the second grey quark and a_4 with the second black one. A value r_i is given to each segment on the left of the cross, and $r_i - a_j$ to the right of the index a_j. Sums over the indices r_i are introduced, with r_i going through the values

$$r_i = 0 - \epsilon, \pm 1 - \epsilon, \pm 2 - \epsilon, \ldots \tag{6.57}$$

and the diagram is given the sign of $(-1)^l$, where l is the number of points at which two segments of the same color cross each other.

The symmetry properties (6.48) and (6.49) imply that the order of the quarks in each nucleon can be changed. Equation (6.50) leads to invariance under exchange of nucleons on the same side (left or right). The domain of the wave function Ω^h are those points r_i in the set of equation (6.57); and if the sum of the four arguments r_i is not zero, Ω^h vanishes; as a consequence, the indices a_i of a diagram must be integers and if their sum is not zero, the diagram is zero

$$a_1,\ a_2,\ a_3,\ a_4, \text{ integers} \tag{6.58}$$

$$\mathcal{D}_j^h(a_1, a_2, a_3, a_4) = \delta(a_1 + a_2 + a_3 + a_4)\ \mathcal{D}_j^h(a_1, a_2, a_3, a_4). \tag{6.59}$$

Equation (6.51) leads to invariance of the diagrams under reflection from left to right; also

$$\mathcal{D}_j^h(a_1, a_2, a_3, a_4) = \mathcal{D}_j^h(-a_1, -a_2, -a_3, -a_4). \tag{6.60}$$

We will now find an analytic form for the diagrams. If we replace Ω^h in terms of equation (6.47), we have

$$\begin{aligned}\mathcal{D}_d^h(a_1, a_2, a_3, a_4) = 2 \sum_{(r_i+\epsilon)_{i=1\ldots4}} & \delta(r_1 + r_2 + r_3 + r_4)\ \omega_h(r_1 + r_2)\ \omega_h(r_1 + r_2 - a_1 - a_2) \\ \times\ \omega\left(\frac{r_1 - r_2}{2}\right)\ & \omega\left(\frac{r_1 - r_2 - a_1 + a_2}{2}\right)\ \omega\left(\frac{r_3 - r_4}{2}\right)\ \omega\left(\frac{r_3 - r_4 - a_3 + a_4}{2}\right);\end{aligned} \tag{6.61}$$

$r_2 - r_3$ is always an integer, but if it is not even then the function $\omega((r_2 - r_4)/2)$ will vanish; hence we consider only $r_2 - r_3$ even and p_1 is thus an integer; but if $r_2 - r_3$ is even then the parity of $r_2 + r_3$ is the same as that of 2ϵ and therefore $(r_2 + r_3)/2$ gives an integer (p_3) plus ϵ. For r_1 and r_3 a similar argument holds. Therefore, the rational dummy indices can be replaced by the indices

$$p_1 \equiv \frac{r_1 - r_2}{2}, \qquad p_2 \equiv \frac{r_3 - r_4}{2}, \tag{6.62}$$

$$p_3 + \epsilon \equiv \frac{r_1 + r_2}{2}, \qquad p_4 + \epsilon \equiv \frac{r_3 + r_4}{2}, \tag{6.63}$$

which are all integers. With this change of indices and by evaluating the sum over p_4 via the delta function, we can write the direct diagram in the form

$$\begin{aligned} \mathcal{D}_d^h(a_1,a_2,a_3,a_4) &= 2 \sum_{p_1,p_2,p_3} \omega_h(2p_3+2\epsilon)\, \omega_h(2p_3+2\epsilon-a_1-a_2)\, \omega(p_1) \\ &\quad \times \omega\left(p_1+\frac{a_2-a_1}{2}\right)\, \omega(p_2)\, \omega\left(p_2+\frac{a_4-a_3}{2}\right). \end{aligned} \tag{6.64}$$

With the algebraic form of Ω^h, the exchange diagram becomes

$$\begin{aligned} \mathcal{D}_e^h(a_1,a_2,a_3,a_4) = -2 \sum_{(r_i+\epsilon)_{i=1\ldots4}} & \delta(r_1+r_2+r_3+r_4)\, \omega_h(r_1+r_2) \\ \times\, \omega_h(r_1+r_3-a_1-a_3)\, \omega\left(\frac{r_1-r_2}{2}\right)\, & \omega\left(\frac{r_1-r_3-a_1+a_3}{2}\right)\, \omega\left(\frac{r_3-r_4}{2}\right)\, \omega\left(\frac{r_2-r_4-a_2+a_4}{2}\right). \end{aligned} \tag{6.65}$$

The only points that contribute to the sum are those for which the arguments of the four functions ω are integers; this implies that the numbers p_i, defined as

$$p_1 \equiv \frac{r_1 + r_4 - a_3 + a_1}{2} + \epsilon, \qquad p_2 \equiv \frac{r_1 + r_2}{2} + \epsilon, \tag{6.66}$$

$$p_3 \equiv \frac{r_1 + r_3 - a_1 + a_3}{2} + \epsilon, \qquad p_4 \equiv r_1 + \epsilon,$$

are all integers; moreover, for every point (r_1, r_2, r_3, r_4) there is one and only one corresponding point (p_1, p_2, p_3, p_4), and the sums can be replaced by

$$\begin{aligned} \mathcal{D}_e^h(a_1,a_2,a_3,a_4) = -2 \sum_{p_1,p_2,p_3,p_4} & \delta(p_1+p_2+p_3-p_4-2\epsilon-a_3-a_1)\, \omega_h(2p_2-2\epsilon) \\ \times\, \omega_h(2p_3-2\epsilon-2a_1)\, & \omega(p_2-p_4)\, \omega(p_3-p_4)\, \omega(p_1-p_3)\, \omega(p_1-p_2+a_2+a_3); \end{aligned} \tag{6.67}$$

in the argument of the last function ω we made use of the fact that the sum of a_i is zero, to get a simpler expression. The sum over p_4 can be evaluated with the delta function, giving the result

$$\mathcal{D}_e^h(a_1,a_2,a_3,a_4) = -2 \sum_{p_1,p_2,p_3} \omega_h(2p_2-2\epsilon)\,\omega_h(2p_3-2\epsilon-2a_1) \qquad (6.68)$$
$$\times\ \omega(p_1+p_3-2\epsilon+a_3-a_1)\,\omega(p_1+p_2-2\epsilon+a_3-a_1))\,\omega(p_1-p_3)\,\omega(p_1-p_2+a_2+a_3),$$

and if the functions ω and ω_h are replaced by their functional form, equations (6.16) and (6.2), we have

$$\mathcal{D}_e^h = -2\left(\frac{\beta^2\beta_h}{4}\right)^2 \sum_{p_1,p_2,p_3} \left[(p_2-\epsilon)^2 - K^2\right]^{-1} \left[(p_3-\epsilon-a_1)^2 - K^2\right]^{-1} \left[(p_1+p_3-2\epsilon-c_1)^2+\alpha^2\right]^{-1}$$
$$\times \left[(p_1+p_2-2\epsilon-c_1)^2+\alpha^2\right]^{-1} \left[(p_3-p_1)^2+\alpha^2\right]^{-1} \left[(p_2-p_1-c_2)^2+\alpha^2\right]^{-1}; \qquad (6.69)$$

the integers c_1 and c_2 have been defined as

$$c_1 \equiv a_1 - a_3 \quad ; \quad c_2 \equiv a_2 + a_3. \qquad (6.70)$$

Finally, we will write the sums in terms of functions ξ

$$\mathcal{D}_e^h(a_1,a_2,a_3,a_4) = -2\left(\frac{\beta^2\beta_h}{4}\right)^2 \sum_p \xi_6\left(\epsilon+a_1\pm K\ ,\ 2\epsilon-p+c_1\pm i\alpha\ ,\ p\pm i\alpha\right)$$
$$\times\xi_6\left(\epsilon\pm K\ ,\ 2\epsilon-p+c_1\pm i\alpha\ ,\ p+c_2\pm i\alpha\right). \qquad (6.71)$$

The functions ξ_6 are defined analytically in appendix A; but there is still one series that must be summed. The sum is done in appendix B and the final result is given there.

6.2.2 Direct terms in the cluster approximation

We will first calculate the direct terms of the quark functions, and show that all of them give the same results as the impulse approximation. We first have to normalize the states; according to equation (2.84), the direct part of the norm is

$$\langle\Omega^h\,|\,\Omega^h\rangle_d = \frac{1}{2}\mathcal{D}_d^h(0,0,0,0); \qquad (6.72)$$

and from the general expression for the direct diagram, we can then write the norm as

$$\langle\Omega^h\,|\,\Omega^h\rangle_d = \sum_{p_3} \omega_h^2(2p_3+2\epsilon) \sum_{p_1} \omega^2(p_1) \sum_{p_2} \omega^2(p_2) = 1; \qquad (6.73)$$

the result of 1 has been obtained because the hadronic and quark states have already been normalized.

The elastic form factor is obtained from equation (2.88); there are four direct terms which lead to the same diagram

$$F_d^h(q) == \frac{1}{2}\mathcal{D}_d^h(-3q/4, q/4, q/4, q/4); \tag{6.74}$$

since the arguments of the diagram must be integers, the momentum transfer must be a multiple of four. We have

$$F_d^h(q) = \sum_{p_1,p_2,p_3} \omega_h(2p_3+2\epsilon)\,\omega_h(2p_3+2\epsilon-2p)\,\omega^2(p_1)\,\omega(p_2)\,\omega(p_2+2p); \tag{6.75}$$

here $p \equiv q/4$ is an integer. Since the state has been normalized, the sum over p_1 is equal to one and we have

$$F_d^h(q) = \sum_{p_1} \omega_h(2p_1+2\epsilon)\,\omega_h(2p_1+2\epsilon-2p) \sum_{p_2} \omega(p_2)\,\omega(p_2+2p). \tag{6.76}$$

Comparing with equation (6.7) we see that the sum over p_2 gives the elastic form factor of the nucleon, $f(q)$. The sum over p_1 is $f_h(q)$ according to equation (6.29); therefore this result is exactly the same as the impulse approximation result.

The direct term of the same–color correlation is obtained from equation (2.99)

$$\begin{aligned} C_d^h(q) &= \frac{1}{2}\mathcal{D}_d^h(-q, 0, q, 0), \\ &= \sum_{p_1,p_2,p_3} \omega_h(2p_3+2\epsilon)\,\omega_h(2p_3+2\epsilon+q)\,\omega(p_1)\,\omega(p_1+q/2)\,\omega(p_2)\,\omega(p_2-q/2). \end{aligned} \tag{6.77}$$

The sums over p_1 and p_2 are both equal to the elastic form factor of the nucleon, $f(q)$, and the sum over p_3 is the hadron correlation function of the nucleus in the hadron picture, $c_h(q)$; therefore, the direct term of the same–color correlation function in the cluster approximation is identical to the impulse approximation result

$$C_d^h(q) = C_{IA}(q). \tag{6.78}$$

The direct term of the different–color correlation function is given by equation (2.98)

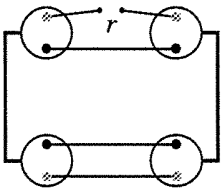
$$D_d^h(q) = \frac{1}{4}\left[\mathcal{D}_d^h(-q, q, 0, 0) + \mathcal{D}_d^h(-q, 0, 0, q)\right],$$

$$= \frac{1}{2}\sum_{p_1,p_2,p_3}\Big[\omega_h^2(2p_3+2\epsilon)\,\omega^2(p_1)\,\omega(p_2)\,\omega(p_2-q)$$
$$+ \omega_h(2p_3+2\epsilon)\,\omega_h(2p_3+2\epsilon+q)\,\omega(p_1)\,\omega(p_1+q/2)\,\omega(p_2)\,\omega(p_2-q/2)\Big], \quad (6.79)$$

which is identical to the impulse approximation result

$$D_d^h(q) = D_{IA}(q). \quad (6.80)$$

The momentum distribution does not have the same general form as the elastic form factor and correlation functions, but it can also be calculated in terms of diagrams. From equation (2.83); we obtain four direct diagrams for $N_h^h(r)$, which are all topologically equivalent; thus

$$N_d^h(r) = \frac{1}{2}$$ 

$$= \frac{1}{2}\sum_{(r_i+\epsilon)_{i=2,3,4}}\Big[\Omega^h(r,r_2,r_3,r_4)\Big]^2, \quad (6.81)$$

where r is in the same domain as in equation (6.35); substituting equation (6.47), N_d^h becomes

$$N_d^h(r) = \sum_{(r_i+\epsilon)_{i=2,3,4}}\delta(r+r_2+r_3+r_4)\,\omega_h^2(r_3+r_4)\,\omega^2\left(\frac{r_3-r_4}{2}\right)\,\omega^2\left(\frac{r-r_2}{2}\right); \quad (6.82)$$

since r, r_2, r_3 and r_4 are all integers minus ϵ, then $r_3 - r_4$ and $r - r_2$ are integers; moreover, we can assume both integers to be even because the functions $\omega(\frac{r_3-r_4}{2})$ and $\omega(\frac{r-r_2}{2})$ would be zero otherwise; therefore, we can relabel $(r_3 - r_4)/2$ as the integer dummy index p_2, and $(r - r_2)/2$ as the integer $-p_3$. Also, if $r_3 - r_4$ is an even integer then $r_3 + r_4$ is an even integer plus 2ϵ, and $(r_3 + r_4)/2$ can be replaced by $p_1 + \epsilon$ with p_1 being an integer dummy index. These changes lead to

$$N_d^h(r) = \sum_{p_1,p_2,p_3}\delta(r+p_1-p_3+\epsilon)\,\omega_h^2(2p_1+2\epsilon)\,\omega^2(p_2)\,\omega^2(p_3); \quad (6.83)$$

the sum over p_2 is simply the norm of the state of the nucleon, which has been normalized to 1; the sum over p_1 can be evaluated with the delta function and we are left with

$$N_d^h(r) = \sum_p \omega^2(p)\,\omega_h^2(2r-2p), \quad (6.84)$$

which is exactly the same result obtained with the impulse approximation, equation (6.34).

6.2.3 Exchange terms in the cluster approximation

The exchange term of the norm, according to equation (2.84), is

$$\begin{aligned}\langle\Omega^h|\Omega^h\rangle_e &= \frac{1}{2}\mathcal{D}_e^h(0,0,0,0),\\ &= -\left(\frac{\beta^2\beta_h}{4}\right)^2\sum_p \xi_6^2\left(\epsilon\pm K\ ,\ 2\epsilon-p+\pm i\alpha\ ,\ p\pm i\alpha\right).\end{aligned} \tag{6.85}$$

The result is not 1, as was the case for the direct part. Therefore, the observables that we want to calculate wil be obtained by adding the direct term, which is impulse approximation result, plus the exchange term, and the total result should be divided by the norm:

$$\langle\Omega^h|\Omega^h\rangle = 1 + \langle\Omega^h|\Omega^h\rangle_e. \tag{6.86}$$

From equations (2.88), (2.98) and (2.99), the exchange terms of the elastic form factor and correlation functions can be written as the following diagrams

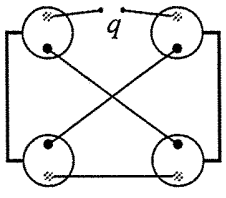
$$F_e^h(q) = \frac{1}{2}\mathcal{D}_e^h(-3q/4, q/4, q/4, q/4), \tag{6.87}$$

$$C_e^h(q) = \frac{1}{2}\mathcal{D}_e^h(0, q, -q, 0), \tag{6.88}$$

$$D_e^h(q) = \frac{1}{2}\mathcal{D}_e^h(-q, q, 0, 0); \tag{6.89}$$

notice that the different-color correlation function, whose direct term had two distinct diagrams, has only one topologically distinct exchange diagram. The analytical form of the exchange terms of the functions are obtained from the analytical form for the exchange diagram, given in appendix B.

To calculate the exchange term of the momentum distribution, we use the diagram

$N_e^h(r) = \frac{1}{2}$ 

$= -\frac{1}{2}\sum_{(r_i+\epsilon)_{i=2,3,4}} \Omega^h(r,r_2,r_3,r_4)\,\Omega^h(r,r_3,r_2,r_4);$ (6.90)

substitution of the wave function Ω^h gives

$$N_e^h(r) = -\frac{1}{2}\sum_{(r_i+\epsilon)_{i=2,3,4}} \Omega^h(r,r_2,r_3,r_4)\,\Omega^h(r,r_3,r_2,r_4); \tag{6.91}$$

from equation (6.47) we have

$$N_e^h(r) = -\sum_{(r_i+\epsilon)_{i=2,3,4}} \delta(r+r_2+r_3+r_4) \tag{6.92}$$
$$\times\, \omega_h(r_2+r_4)\, \omega_h(r_3+r_4)\, \omega\left(\frac{r_2-r_4}{2}\right)\, \omega\left(\frac{r_3-r_4}{2}\right)\, \omega\left(\frac{r-r_3}{2}\right)\, \omega\left(\frac{r-r_2}{2}\right);$$

since $r_2 - r$, $r_3 - r$ and $r_4 - r$ are all integers, and the term inside the sum becomes zero if either $(r_2 - r)/2$, $(r_3 - r)/2$ or $(r_4 - r)/2$ are not integers, then we can relabel $(r_2 - r)/2$, $(r_3 - r)/2$ and $(r_4 - r)/2$ as the integer dummy indices p_2, p_3 and p_1 respectively

$$N_e^h(r) = -\sum_{p_1,p_2,p_3} \delta(2r+p_1+p_2+p_3)$$
$$\times\, \omega_h(2p_1+2p_2+2r)\, \omega_h(2p_1+2p_3+2r)\, \omega(p_2)\, \omega(p_3)\, \omega(p_1-p_2)\, \omega(p_1-p_3); \tag{6.93}$$

evaluation of the sum over p_1, by means of the delta function,and the use of the algebraic form of the functions ω and ω_h lead to

$$N_e^h(r) = -\beta^4\, \beta_h^2 \sum_{p_2,p_3} (p_2^2+\alpha^2)^{-1}(p_3^2+\alpha^2)^{-1} \left[(2p_2+p_3+2r)^2+\alpha^2\right]^{-1} \tag{6.94}$$
$$\times\, \left[(p_2+2p_3+2r)^2+\alpha^2\right]^{-1} \left[(2p_2+2r)^2-k^2\right]^{-1} \left[(2p_3+2r)^2-k^2\right]^{-1};$$

the sums can be arranged so that the sum over p_3 is evaluated first and the terms which do not depend on p_3 are taken out of the sum over p_3; this gives a sum over p_3 that can be written as a function ξ_8

$$N_e^h(r) = -\frac{\beta^4\, \beta_h^2}{64} \sum_p (p^2+\alpha^2)^{-1} \left[(p+r)^2-K^2\right]^{-1}$$
$$\times \xi_8\left(\pm i\alpha, r \pm K, 2r+2p \pm i\alpha, r+(p \pm i\alpha)/2\right), \tag{6.95}$$

where $K \equiv k/2$. The sum over p is more difficult to calculate because we must first find the p dependence of the function ξ_8; the algebra has been carried out in appendix B and the final result is found there.

6.3 Quark model

In the quark model the observables are calculated in exactly the same way as in the cluster approximation. The only difference will be that the exact wave function Ω will be used

instead of the cluster approximation function. This will lead to different analytic expressions for the diagrams. In this section we will derive those expressions. The problem of how to relate the observables to the different diagrams will not be considered, since it has already been discussed in the previous section.

The states of the system in the quark model have been given in chapter 5 (equations (5.11) through (5.13)). In order to make the diagrammatic notation simpler, we will rewrite them in the form

$$|\Omega\rangle = \frac{1}{\sqrt{4}} \sum_{(r_i+\epsilon)_{i=1\ldots4}} \Omega(r_1,r_2,r_3,r_4)\, b^\dagger_{r_1} a^\dagger_{r_2} a^\dagger_{r_3} b^\dagger_{r_4} |0\rangle, \tag{6.96}$$

where the wave function is

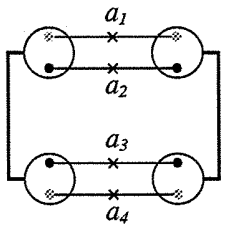
$$\Omega(r_1,r_2,r_3,r_4) = B \sum_{\nu\in S_4} \frac{(-1)^\nu \delta\,(r_1+r_2+r_3+r_4)}{(r_1-K_{\nu_1})(r_4-K_{\nu_4})} \left[\frac{K_{\nu_2}}{r_2-K_{\nu_2}} - \delta_\nu \frac{K_{\nu_2}+K_{\nu_3}}{r_2+r_3-K_{\nu_2}-K_{\nu_3}}\right], \tag{6.97}$$

and the number δ_ν is now defined as

$$\delta\nu = \begin{cases} 0, \text{ if } K_{\nu_2} + K_{\nu_3} = 0 \\ 1, \text{ otherwise.} \end{cases} \tag{6.98}$$

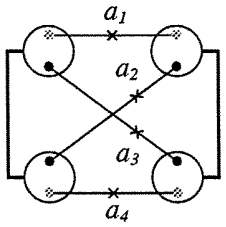
The direct and exchange diagrams are defined in analogy with the cluster approximation

$$\mathcal{D}_d(a_1,a_2,a_3,a_4) = \tag{6.99}$$



$$= \sum_{(r_i+\epsilon)_{i=1,2,3,4}} \Omega(r_1,r_2,r_3,r_4)\, \Omega(r_1-a_1,r_2-a_2,r_3-a_3,r_4-a_4),$$

$$\mathcal{D}_e(a_1,a_2,a_3,a_4) = \tag{6.100}$$



$$= -\sum_{(r_i+\epsilon)_{i=1,2,3,4}} \Omega(r_1,r_2,r_3,r_4)\, \Omega(r_1-a_1,r_3-a_2,r_2-a_3,r_4-a_4),$$

where the numbers a_i are all integers and their sum must be equal to zero. The symmetry properties of the wave function Ω derived at the end of chapter 4, lead to the same topological properties as in the cluster approximation. However, the invariance under exchange of clusters in the diagram is valid only if it is done both in the direct and exchange diagrams; this does not represent any problem, since when the observables are calculated, for each direct diagram there is a corresponding exchange one.

Replacing the wave function Ω by its algebraic form, equation (6.97), in the direct diagram gives

$$\mathcal{D}_d = B^2 \sum_{p_1,p_2,p_3,p_4} \left\{ \sum_{\mu\in S_4} \frac{(-1)^\mu \delta(p_1+p_2+p_3+p_4-4\epsilon)}{(p_1-\widehat{K}_{\mu_1})(p_4-\widehat{K}_{\mu_4})} \left[\frac{K_{\mu_2}}{p_2-\widehat{K}_{\mu_2}} - \frac{K_{\mu_1}+K_{\mu_2}}{p_1+p_2-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2}} \right] \right\}$$
$$\times \left\{ \sum_{\nu\in S_4} \frac{(-1)^\nu \delta(p_1+p_2+p_3+p_4-4\epsilon)}{(p_1-a_1-\widehat{K}_{\nu_1})(p_4-a_4-\widehat{K}_{\nu_4})} \left[\frac{K_{\nu_2}}{p_2-a_2-\widehat{K}_{\nu_2}} - \frac{K_{\nu_1}+K_{\nu_2}}{p_1+p_2-a_1-a_2-\widehat{K}_{\nu_1}-\widehat{K}_{\nu_2}} \right] \right\};$$

the rational dummy indices r_i have been replaced by $p_i - \epsilon$ and the sum runs now over the integer indices p_i; since every denominator in the wave function Ω contains only factors of the form $r_i - K_{\nu_i}$, then when r_i were replaced by $p_i - \epsilon$ the number ϵ was included in the Bethe momenta by defining

$$\widehat{K}_i \equiv K_i + \epsilon \qquad (i = 1,2,3,4). \tag{6.101}$$

We now do the product of the two sums over the symmetric group

$$\mathcal{D}_d = B^2 \sum_{\mu,\nu\in S_4} \sum_{(p_i)_{i=1\ldots4}} \frac{(-1)^{\mu+\nu}\delta(p_1+p_2+p_3+p_4-4\epsilon)}{(p_1-\widehat{K}_{\mu_1})(p_1-\widetilde{K}_{\nu_1})(p_4-\widehat{K}_{\mu_4})(p_4-\widetilde{K}_{\nu_4})}$$
$$\times \left[\frac{K_{\mu_2}K_{\nu_2}}{(p_2-\widehat{K}_{\mu_2})(p_2-\widetilde{K}_{\nu_2})} - \frac{K_{\mu_2}(K_{\nu_1}+K_{\nu_2})}{(p_2-\widehat{K}_{\mu_2})(p_1+p_2-\widetilde{K}_{\nu_1}-\widetilde{K}_{\nu_2})} \right. \tag{6.102}$$
$$\left. - \frac{K_{\nu_2}(K_{\mu_1}+K_{\mu_2})}{(p_2-\widetilde{K}_{\nu_2})(p_1+p_2-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2})} + \frac{(K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})}{(p_1+p_2-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2})(p_1+p_2-\widetilde{K}_{\nu_1}-\widetilde{K}_{\nu_2})} \right];$$

the numbers a_i have also been included in the Bethe momenta K_{ν_i}, in a similar way as we did with ϵ, by introducing the following notation

$$\widetilde{K}_{\nu_i} \equiv \widehat{K}_{\nu_i} + a_i = K_{\nu_i} + a_i + \epsilon. \tag{6.103}$$

The sum over p_3 is trivial because the only p_3 dependence is in the delta function (and 4ϵ is an integer). The remaining sums can be written in terms of the functions ξ_2 and S_2, whose

definitions, together with an analytic scheme to calculate them are given in appendix A; therefore, we will leave G in the final form

$$\begin{aligned}\mathcal{D}_d = B^2 \sum_{\nu,\mu\in S_4} (-1)^{\mu+\nu}\xi_2(\widehat{K}_{\mu_4},\widetilde{K}_{\nu_4})\Big[K_{\mu_2}K_{\nu_2}\;\xi_2(\widehat{K}_{\mu_1},\widetilde{K}_{\nu_1})\;\xi_2(\widehat{K}_{\mu_2},\widetilde{K}_{\nu_2})\\ -K_{\mu_2}(K_{\nu_1}+K_{\nu_2})\;S_2(\widehat{K}_{\mu_1},\widetilde{K}_{\nu_1},\widehat{K}_{\mu_2},\widetilde{K}_{\nu_1}+\widetilde{K}_{\nu_2})\\ -K_{\nu_2}(K_{\mu_1}+K_{\mu_2})\;S_2(\widehat{K}_{\mu_1},\widetilde{K}_{\nu_1},\widetilde{K}_{\nu_2},\widehat{K}_{\mu_1}+\widehat{K}_{\mu_2})\\ +(K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})\;\xi_2(\widehat{K}_{\mu_1},\widetilde{K}_{\nu_1})\;\xi_2(\widehat{K}_{\mu_1}+\widehat{K}_{\mu_2},\widetilde{K}_{\nu_1}+\widetilde{K}_{\nu_2})\Big]\,.\end{aligned} \tag{6.104}$$

We will now compute the exchange diagram; using the form of the wave function Ω, we have

$$\begin{aligned}\mathcal{D}_e = -B^2 \sum_{p_1,p_2,p_3,p_4} \delta(p_1+p_2+p_3+p_4-4\epsilon)\\ \times\left\{\sum_{\mu\in S_4}\frac{(-1)^\mu}{(p_1-\widehat{K}_{\mu_1})(p_4-\widehat{K}_{\mu_4})}\left[\frac{K_{\mu_2}}{p_2-\widehat{K}_{\mu_2}}-\frac{K_{\mu_1}+K_{\mu_2}}{p_1+p_2-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2}}\right]\right\}\\ \times\left\{\sum_{\nu\in S_4}\frac{(-1)^\nu}{(p_1-a_1-\widehat{K}_{\nu_1})(p_4-a_4-\widehat{K}_{\nu_4})}\left[\frac{K_{\nu_2}}{p_3-a_2-\widehat{K}_{\nu_2}}-\frac{K_{\nu_1}+K_{\nu_2}}{p_1+p_3-a_1-a_2-\widehat{K}_{\nu_1}-\widehat{K}_{\nu_2}}\right]\right\};\end{aligned} \tag{6.105}$$

the rational dummy indices were replaced by integers, and K_{ν_i} was replaced by $\widehat{K}_{\nu_i}$ as we did in the direct term. The product of the expressions inside the curly brackets leads to

$$\begin{aligned}\mathcal{D}_e = -B^2 \sum_{\mu,\nu\in S_4}\sum_{(p_i)_{i=1\ldots4}} \frac{(-1)^{\mu+\nu}\delta(p_1+p_2+p_3+p_4-4\epsilon)}{(p_1-\widehat{K}_{\mu_1})(p_1-\widetilde{K}_{\nu_1})(p_4-\widehat{K}_{\mu_4})(p_4-\widetilde{K}_{\nu_4})}\\ \times\left[\frac{K_{\mu_2}K_{\nu_2}}{(p_2-\widehat{K}_{\mu_2})(p_3-\widetilde{K}_{\nu_2})}-\frac{K_{\mu_2}(K_{\nu_1}+K_{\nu_2})}{(p_2-\widehat{K}_{\mu_2})(p_1+p_3-\widetilde{K}_{\nu_1}-\widetilde{K}_{\nu_2})}\right.\\ \left.-\frac{K_{\nu_2}(K_{\mu_1}+K_{\mu_2})}{(p_2-\widetilde{K}_{\nu_2})(p_1+p_3-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2})}+\frac{(K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})}{(p_1+p_2-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2})(p_1+p_3-\widetilde{K}_{\nu_1}-\widetilde{K}_{\nu_2})}\right];\end{aligned} \tag{6.106}$$

in this last step the notation of equation (6.103) was introduced; notice that in the third term inside the square brackets the dummy indices p_2 and p_3 were exchanged, in order to make the next step simpler. Unlike the direct term, p_3 appears in all of the denominators; however the sum over it is still easily calculated by means of the delta function; furthermore, we will eliminate ϵ from our result as follows: the Bethe momenta add up to zero; therefore, for any permutation ν we have

$$K_{\nu_1}+K_{\nu_2}+K_{\nu_3}+K_{\nu_4}=0, \tag{6.107}$$

which implies

$$\widehat{K}_{\nu_1} + \widehat{K}_{\nu_2} + \widehat{K}_{\nu_3} + \widehat{K}_{\nu_4} = 4\epsilon, \tag{6.108}$$

and since the numbers a_i add up to zero, then

$$\widetilde{K}_{\nu_1} + \widetilde{K}_{\nu_2} + \widetilde{K}_{\nu_3} + \widetilde{K}_{\nu_4} = 4\epsilon. \tag{6.109}$$

These last two relations can be used to substitute ϵ in terms of the Bethe momenta, and thus the evaluation of the sum over p_3 leads to

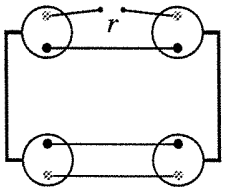
$$\begin{aligned}\mathcal{D}_e = B^2 \sum_{\mu,\nu\in S_4} \sum_{p_1,p_2,p_4} & \frac{(-1)^{\mu+\nu}}{(p_1-\widehat{K}_{\mu_1})(p_1-\widetilde{K}_{\nu_1})(p_4-\widehat{K}_{\mu_4})(p_4-\widetilde{K}_{\nu_4})} \\ \times \Bigg[& \frac{K_{\mu_2}K_{\nu_2}}{(p_2-\widehat{K}_{\mu_2})((p_1+p_2+p_4--\widetilde{K}_{\nu_1}-\widetilde{K}_{\nu_3}-\widetilde{K}_{\nu_4})} \\ & -\frac{K_{\mu_2}(K_{\nu_1}+K_{\nu_2})}{(p_2-\widehat{K}_{\mu_2})(p_2+p_4-\widetilde{K}_{\nu_3}-\widetilde{K}_{\nu_4})} - \frac{K_{\nu_2}(K_{\mu_1}+K_{\mu_2})}{(p_2-\widetilde{K}_{\nu_2})(p_2+p_4-\widehat{K}_{\mu_3}-\widehat{K}_{\mu_4})} \\ & +\frac{(K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})}{(p_1+p_2-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2})(p_2+p_4-\widetilde{K}_{\nu_3}-\widetilde{K}_{\nu_4})}\Bigg];\end{aligned} \tag{6.110}$$

and finally we will write down the remaining sums in terms of the functions ξ and S introduced in appendix A

$$\begin{aligned}\mathcal{D}_e = B^2 \sum_{\mu,\nu\in S_4} (-1)^{\mu+\nu} \Big[& K_{\mu_2}K_{\nu_2}\, S_3(\widehat{K}_{\mu_1},\widetilde{K}_{\nu_1},\widehat{K}_{\mu_4},\widetilde{K}_{\nu_4},\widehat{K}_{\mu_2},\widetilde{K}_{\nu_1}+\widetilde{K}_{\nu_3}+\widetilde{K}_{\nu_4}) \\ & -K_{\mu_2}(K_{\nu_1}+K_{\nu_2})\,\xi_2(\widehat{K}_{\mu_1},\widetilde{K}_{\nu_1})\,S_2(\widehat{K}_{\mu_4},\widetilde{K}_{\nu_4},\widehat{K}_{\mu_2},\widetilde{K}_{\nu_3}+\widetilde{K}_{\nu_4}) \\ & -K_{\nu_2}(K_{\mu_1}+K_{\mu_2})\,\xi_2(\widehat{K}_{\mu_1},\widetilde{K}_{\nu_1})\,S_2(\widetilde{K}_{\nu_4},\widehat{K}_{\mu_4},\widetilde{K}_{\nu_2},\widehat{K}_{\mu_3}+\widehat{K}_{\mu_4}) \\ & +(K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})\,S_3(-\widehat{K}_{\mu_1},-\widetilde{K}_{\nu_1},\widehat{K}_{\mu_4},\widetilde{K}_{\nu_4},\widehat{K}_{\mu_1}+\widehat{K}_{\mu_2},\widetilde{K}_{\nu_3}+\widetilde{K}_{\nu_4})\Big]\end{aligned} \tag{6.111}$$

The direct diagram for the momentum distribution is

$$N_d(r)\frac{1}{2} = \quad = \frac{1}{2}\sum_{(r_i+\epsilon)_{i=2,3,4}} \Omega^2(r,r_2,r_3,r_4); \tag{6.112}$$



using the functional form of Ω, replacing r_i by $p_i - \epsilon$ and r by $q - \epsilon$ we have

$$N_d(r) = \frac{B^2}{2}\sum_{p_2,p_3,p_4}\left\{\sum_{\nu\in S_4}\frac{(-1)^{\nu}\delta(q+p_2+p_3+p_4-4\epsilon)}{(q-\widehat{K}_{\nu_1})(p_4-\widehat{K}_{\nu_4})}\left[\frac{K_{\nu_2}}{p_2-\widehat{K}_{\nu_2}} - \frac{K_{\nu_1}+K_{\nu_2}}{p_2+q-\widehat{K}_{\nu_1}-\widehat{K}_{\nu_2}}\right]\right\}^2,$$

where q is an integer defined on equation (6.35); p_2, p_3 and p_4 take now integer values, and $\widehat{K}_i$ have been defined as

$$\widehat{K}_i \equiv K_i + \epsilon \qquad (i = 1,2,3,4); \tag{6.113}$$

squaring the expression inside the curly brackets and exchanging the order of the sums lead to

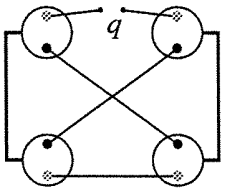
$$\begin{aligned} N_d(r) = \frac{B^2}{2} \sum_{\nu,\mu} \frac{(-1)^{\mu+\nu}}{(q-\widehat{K}_{\mu_1})(q-\widehat{K}_{\nu_1})} \sum_{p_2,p_3,p_4} \frac{\delta(q+p_2+p_3+p_4-4\epsilon)}{(p_4-\widehat{K}_{\mu_4})(p_4-\widehat{K}_{\nu_4})} \\ \times \left[\frac{K_{\mu_2} K_{\nu_2}}{(p_2-\widehat{K}_{\mu_2})(p_2-\widehat{K}_{\nu_2})} - \frac{2K_{\mu_2}(K_{\nu_1}+K_{\nu_2})}{(p_2-\widehat{K}_{\mu_2})(p_2+q-\widehat{K}_{\nu_1}-\widehat{K}_{\nu_2})} \right. \\ \left. + \frac{(K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})}{(p_2+q-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2})(p_2+q-\widehat{K}_{\nu_1}-\widehat{K}_{\nu_2})} \right]; \end{aligned} \tag{6.114}$$

the sum over p_3 is trivial because p_3 appears only inside the delta function; the sums over p_2 and p_4 can be written in terms of the function ξ defined in appendix A; therefore, our final result is

$$\begin{aligned} N_d(r) = \frac{B^2}{2} \sum_{\nu,\mu} \frac{(-1)^{\mu+\nu} \xi_2(\widehat{K}_{\mu_4}, \widehat{K}_{\nu_4})}{(q-\widehat{K}_{\mu_1})(q-\widehat{K}_{\nu_1})} \\ \times \left[K_{\mu_2} K_{\nu_2}\, \xi_2(\widehat{K}_{\mu_2}, \widehat{K}_{\nu_2}) - 2K_{\mu_2}(K_{\nu_1}+K_{\nu_2})\, \xi_2(\widehat{K}_{\mu_2}, \widehat{K}_{\nu_1}+\widehat{K}_{\nu_2}-q) \right. \\ \left. + (K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})\, \xi_2(\widehat{K}_{\mu_1}+\widehat{K}_{\mu_2}, \widehat{K}_{\nu_1}+\widehat{K}_{\nu_2}) \right]. \end{aligned} \tag{6.115}$$

We must point out that $\widehat{K}_{\nu_1} + \widehat{K}_{\nu_2}$ can happen to be an integer in which case ξ_2 in the last two terms is not defined; but in such case $K_{\nu_1} + K_{\nu_2}$ vanishes, and as was explained when Ω was derived in chapter 4, the second term in Ω does not appear when $K_{\nu_1} + K_{\nu_2} = 0$; therefore, each term in N_d must be ignored when one of the arguments of the functions ξ_2 becomes an integer.

The diagram of the momentum distribution in the quark model is

$$N_e(r)\frac{1}{2} = \quad 

\quad = -\frac{1}{2} \sum_{(r_i+\epsilon)_{i=2,3,4}} \Omega(r, r_2, r_3, r_4)\, \Omega(r, r_3, r_2, r_4); \tag{6.116}$$

following the same steps that led to equation (6.114), we obtain a result very similar to equation (6.114) but with p_2 and p_3 exchanged in some of the factors

$$N_e(r) = -\frac{B^2}{2}\sum_{\nu,\mu}\frac{(-1)^{\mu+\nu}}{(q-\widehat{K}_{\mu_1})(q-\widehat{K}_{\nu_1})}\sum_{p_2,p_3,p_4}\frac{\delta(q+p_2+p_3+p_4-4\epsilon)}{(p_4-\widehat{K}_{\mu_4})(p_4-\widehat{K}_{\nu_4})}$$
$$\times\left[\frac{K_{\mu_2}K_{\nu_2}}{(p_2-\widehat{K}_{\mu_2})(p_3-\widehat{K}_{\nu_2})}-\frac{2K_{\mu_2}(K_{\nu_1}+K_{\nu_2})}{(p_2-\widehat{K}_{\mu_2})(p_3+q-\widehat{K}_{\nu_1}-\widehat{K}_{\nu_2})}\right.$$
$$\left.+\frac{(K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})}{(p_2+q-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2})(p_3+q-\widehat{K}_{\nu_1}-\widehat{K}_{\nu_2})}\right]; \tag{6.117}$$

before we evaluate the sum over p_3 using the delta function, let us first notice that for any permutation ν

$$K_{\nu_1}+K_{\nu_2}+K_{\nu_3}+K_{\nu_4}=0, \tag{6.118}$$

and therefore

$$\widehat{K}_{\nu_1}+\widehat{K}_{\nu_2}+\widehat{K}_{\nu_3}+\widehat{K}_{\nu_4}=4\epsilon; \tag{6.119}$$

the use of this result to eliminate ϵ and the evaluation of the sum over p_3 lead to

$$N_e(r) = \frac{B^2}{2}\sum_{\nu,\mu}\sum_{p_2,p_4}\frac{(-1)^{\mu+\nu}}{(r-\widehat{K}_{\mu_1})(r-\widehat{K}_{\nu_1})(p_4-\widehat{K}_{\mu_4})(p_4-\widehat{K}_{\nu_4})}$$
$$\times\left[\frac{K_{\mu_2}K_{\nu_2}}{(p_2-\widehat{K}_{\mu_2})(p_2+p_4+r-\widehat{K}_{\nu_1}-\widehat{K}_{\nu_3}-\widehat{K}_{\nu_4})}-\frac{2K_{\mu_2}(K_{\nu_1}+K_{\nu_2})}{(p_2-\widehat{K}_{\mu_2})(p_2+p_4-\widehat{K}_{\nu_3}-\widehat{K}_{\nu_4})}\right.$$
$$\left.+\frac{(K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})}{(p_2+r-\widehat{K}_{\mu_1}-\widehat{K}_{\mu_2})(p_2+p_4-\widehat{K}_{\nu_3}-\widehat{K}_{\nu_4})}\right]; \tag{6.120}$$

in appendix B it is explained how to evaluate double sums of the kind that we have here, which we denote as S_2; therefore, we will finally leave the result in the following form

$$N_e(r) = \frac{B^2}{2}\sum_{\nu,\mu}\frac{(-1)^{\mu+\nu}}{(r-\widehat{K}_{\mu_1})(r-\widehat{K}_{\nu_1})}\Big[K_{\mu_2}K_{\nu_2}\ S_2(\widehat{K}_{\mu_4},\widehat{K}_{\nu_4},\widehat{K}_{\mu_2},\widehat{K}_{\nu_1}+\widehat{K}_{\nu_3}++\widehat{K}_{\nu_4}-r)$$
$$-2K_{\mu_2}(K_{\nu_1}+K_{\nu_2})\ S_2(\widehat{K}_{\mu_4},\widehat{K}_{\nu_4},\widehat{K}_{\mu_2},\widehat{K}_{\nu_3}+\widehat{K}_{\nu_4})$$
$$+(K_{\mu_1}+K_{\mu_2})(K_{\nu_1}+K_{\nu_2})\ S_2(\widehat{K}_{\mu_4},\widehat{K}_{\nu_4},\widehat{K}_{\mu_1}+\widehat{K}_{\mu_2},\widehat{K}_{\nu_3}+\widehat{K}_{\nu_4}+r)\Big]. \tag{6.121}$$

6.4 Summary

Analytic expressions have been derived for the dynamical functions related to scattering experiments. Cluster diagrams have been used, and all of the functions can be derived from

a few general diagrams. In the quark model the functional form of the diagrams has been written in terms of the functions ξ and S; these functions can be calculated as a finite sum of simple rational functions using the algorithms developed in appendices A and B.

The functional form of the diagrams in the cluster approximation can be found in appendix B. They also involve functions ξ; at first sight it may seem that the results of the cluster approximation are more complicated, but that is not the case —even though the quark model results were written in a more compact form, they involve sums over the symmetric group. The direct diagrams in the cluster approximation lead to the simple results of the impulse approximation, and they constitute a test of the results of section 3.5, which were derived in a very intuitive way.

Since the derivation of the analytic form of the diagrams has required some tedious algebra, it is important to develop some kind of test to make sure the results are correct. The first test comes from the topological properties mentioned above; each diagram must be invariant under exchange of clusters, exchange of quarks inside one cluster, and mirror reflection across the vertical symmetry axis. A second test is provided by the following sum rules

$$\sum_p N_j(p) = \mathcal{D}_j(0,0,0,0), \tag{6.122}$$

$$\sum_q [\mathcal{D}_d(0,q,-q,0) + \mathcal{D}_e(0,q,-q,0)] = 0, \tag{6.123}$$

where the index j stands for direct or exchange diagram. The first sum rule arises from the way the dynamical functions have been normalized; the second one is the statement that the sum of the same-color correlation function must vanish, as required by the exclusion principle. Both sum rules must be satified in the quark model and in the cluster approximation.

The deuteron states have a wave function with terms which have products of three factors in the denominator; each simple factor is the difference between one of the dummy indices p_i and one of the Bethe momenta (or sums of two of such factors). We do not know the form of the deuteron states in the three-color case, but we suspect that it would have a similar general form, with products of five terms rather than three; the analytic expressions for the observables could then be calculated in a similar way as done in the two-color case, but we would encounter also functions S_4 and S_5, in addition to S_2 and S_3. The generalization of the

functions S would be a simple task, following the technique of appendix B. However, even if the results could also be written in a compact form as in this chapter, difficulties would arise when one tries to obtain numerical results; instead of the 288 terms produced by the sums over the group S_4 we would be left with 259,200 terms arising from the permutations of the six quarks in the three-color case. Thus, the computing time would increase by three orders of magnitude. To make the problem manageable, one could try to identify terms which should be identical, using symmetry properties of diagrams; otherwise approximations would have to be used, such as, neglecting certain terms in the sum.

The approximations based on the effective theory are easier to generalize to the three-color case. That is another argument in favor of testing its results in simple models like the one in this dissertation. A detailed, realistic description of the nucleus being very difficult, one may opt to introduce a cluster model or an impulse approximation; however, as we will see in chapter 7, under some conditions these approximations may not be valid.

Chapter 7

Results

In this chapter we show the results obtained for the observables discussed in chapter 6, both in our quark model and the two approximations to it. All momenta will be given in units of $1/R$, where R is the diameter of the nucleon [1]. Some comments will be made about the results obtained; chapter 8 contains also some discussion of the results.

We will concentrate mainly in the cases $L = 2R$ and $L = 5R$ because our estimate for the deuteron is within this range, as we mentioned in chapter 6; the value of $L = 5R$ seems to be the one that is most consistent with the energetics of the deuteron.

7.1 Momentum distribution

The exact momentum distribution $N(p)$ is a monotonically decreasing function, which decays asymptotically as p^{-4}. In the case $L = 5R$ (figure 7-1), it decreases 6 orders of magnitude from $p = 0$ to $p = 20/R$ ($\approx 4GeV$). Figure 7-2 shows the results for the high density case, $L = 2$; at larger values of the parameter L, $N(p)$ decreases faster.

The impulse approximation reproduces remarkably well the exact result; not only in its asymptotic form (p^{-4}), but also at intermediate momentum p. The contribution from exchange terms to the quark model result is very small; that is also the case in the cluster approximation. The results from both approximation methods are very similar.

[1] This choice of units is explained in the section on units in chapter 5

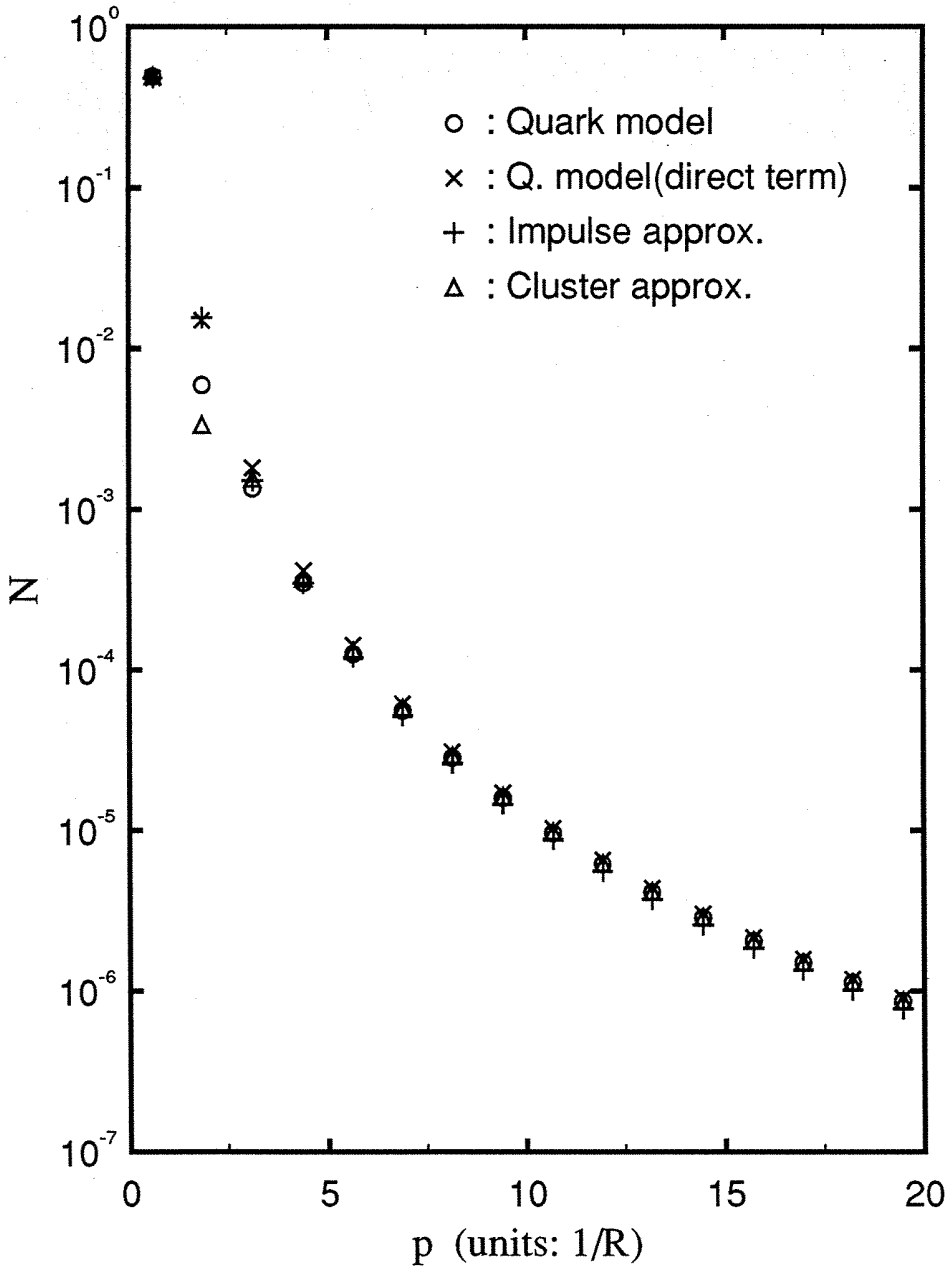


Figure 7-1: Momentum distribution in the quark model and cluster and impulse approximations, for $L = 5R$.

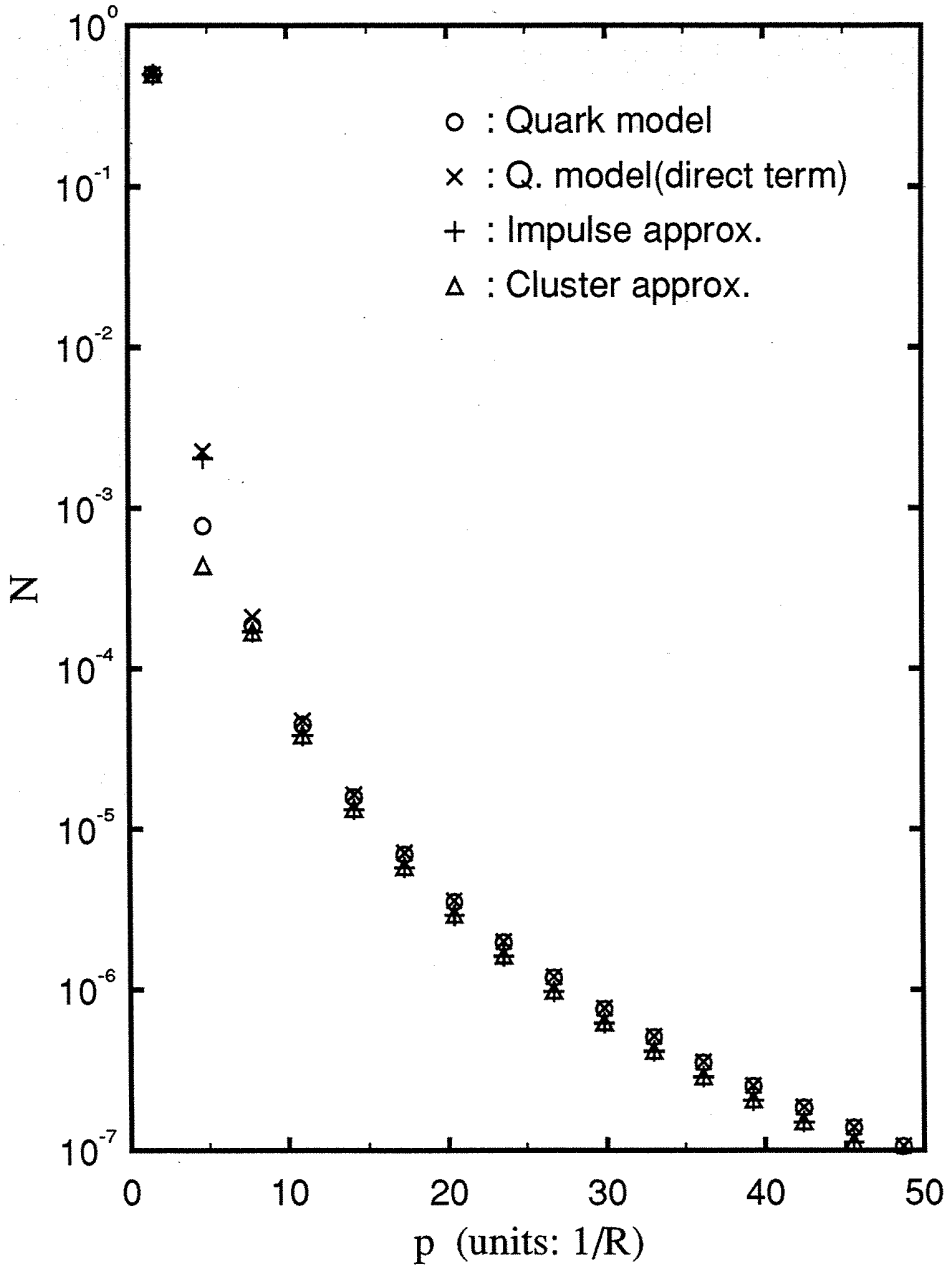


Figure 7-2: Momentum distribution in the quark model and cluster and impulse approximations, for $L = 2R$.

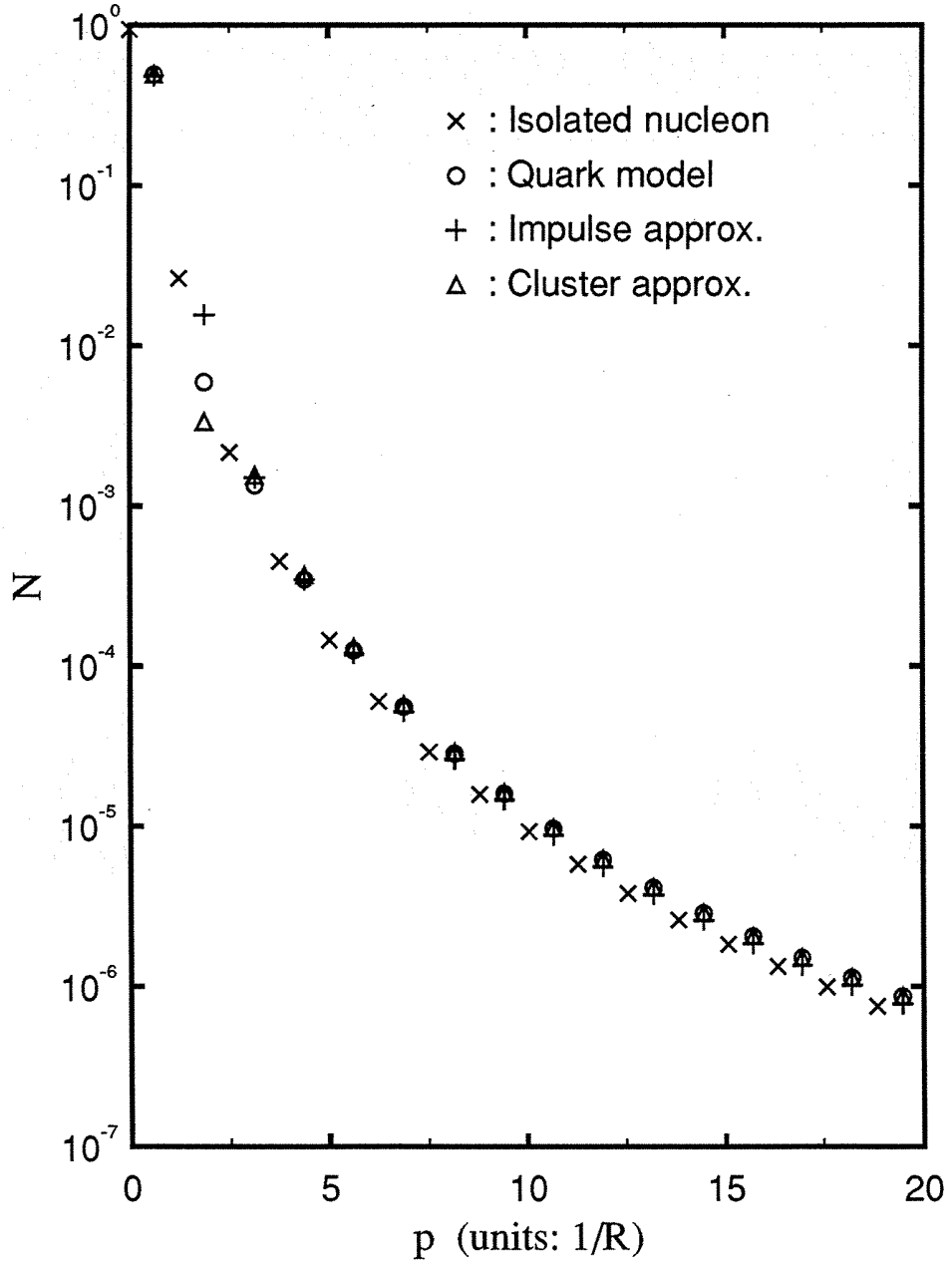


Figure 7-3: Quark momentum distribution in the isolated nucleon, and inside the deuteron (in the case $L = 5R$).

The momentum distribution of the nucleus is almost identical to that of one cluster, as shown in figure 7-3. Therefore, we do not obtain the behavior expected from the EMC effect; i. e. modification of the quark-momentum distribution inside the nuclear medium. However, the momentum distributions derived from the EMC results are given in the *infinite momentum frame*; also, the effect is very small for light targets such as the deuteron. Perhaps this is the reason why the impulse approximation has worked so well. At small momentum we do not have enough points to decide whether the situation is the same. The limitation in the number of points comes from the choice of a finite volume in our model, which is necessary to maintain the clusters bound to the nucleus.

The momentum distribution can be used to calculate the kinetic energy, as explained in chapter 2. The kinetic energy is a one-body operator and in units of b is given by

$$T = \left(\frac{2\pi R}{L}\right)^2 \sum_p p^2 \, (a_p^\dagger a_p + b_p^\dagger b_p); \tag{7.1}$$

using the definition of the momentum distribution, equation (2.36), it can be written as

$$\langle T \rangle = n \left(\frac{2\pi R}{L}\right)^2 \sum_{r+1/2} r^2 N(r). \tag{7.2}$$

Table 7-1 shows the kinetic energy for a few values of the parameter L. To obtain the values in the table, the infinite sum in equation (7.2) was calculated approximately by adding a finite number of points at small momentum r, and beyond that we used the asymptotic form of $N(r)$ which gave us a series that could be summed analytically.

7.2 Different–color correlation function

When L and R are of the same order of magnitude, the different–color correlation function has a minimum at $q = 2\pi/L$, and a maximum at $q = 4\pi/L$. As a consequence, the distance between two quarks with different color is most likely close to $L/2$. Beyond the maximum, $D(q)$ decreases monotonically as q^{-2} (see figure 7-4).

Exchange terms are small compared to direct ones. The impulse and cluster approximations are very close to the exact results as shown in figures 7-4 through 7-6. The agreement is not as good as it was for the momentum distribution, but it is still good.

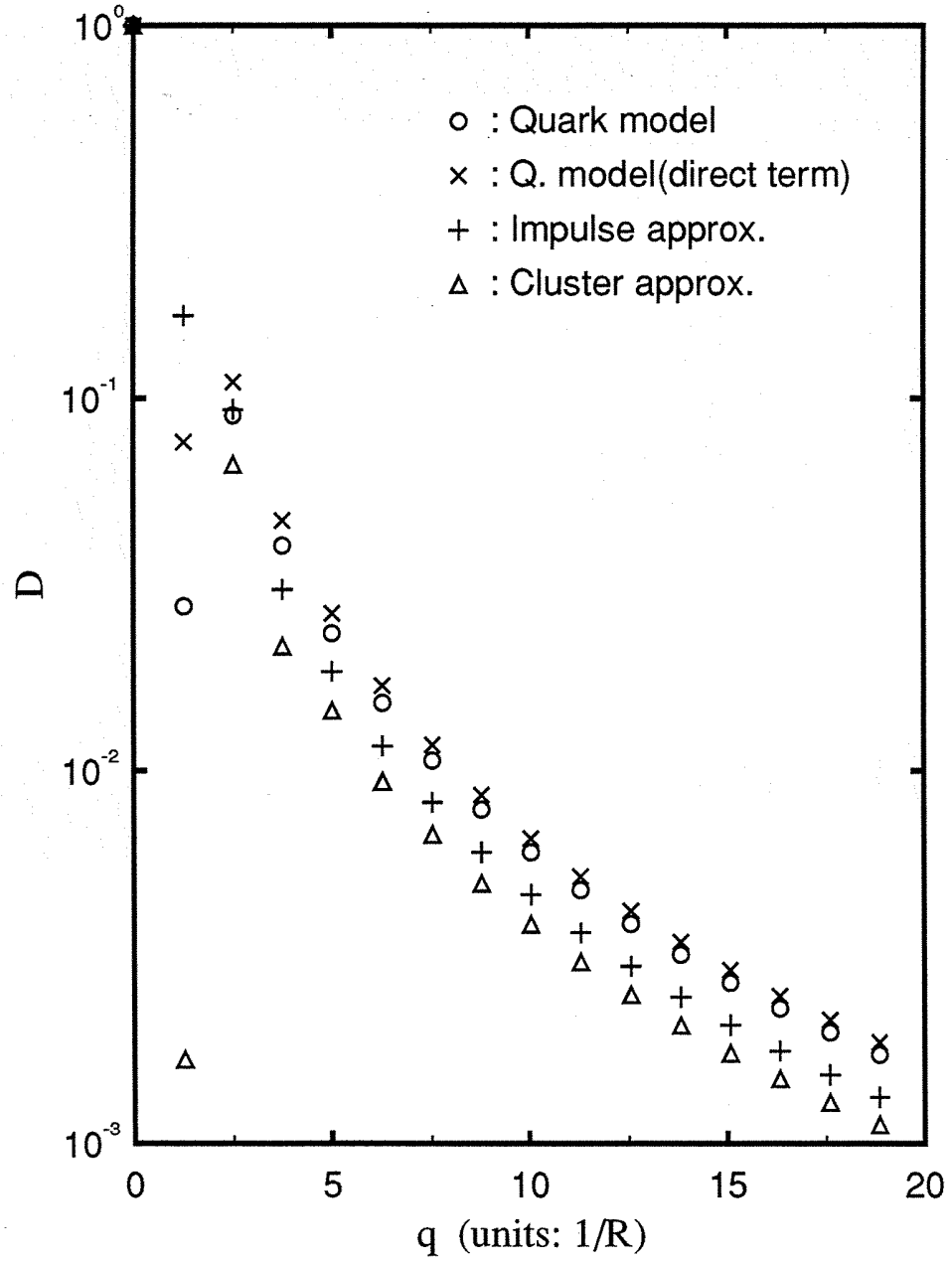


Figure 7-4: Different–color correlation function for $L = 5R$.

L	$\langle T \rangle$	$\langle U \rangle$	$\langle H \rangle$	E_K
2R	10.244	-4.702	5.542	5.511
5R	2.041	-2.453	-0.412	-0.424
10R	1.073	-1.971	-0.898	-0.900
15R	1.033	-1.981	-0.948	-0.950
50R	1.006	-2.006	-1.000	-0.996

Table 7.1: Kinetic, potential and total energies, in the quark model, calculated from the momentum distribution and different–color correlation function. The last column (E_K) is the value obtained directly from the Bethe momentum K. All values are in units of $b(\approx 40MeV)$.

With the different–color correlation function, the expectation value of the potential energy can be calculated as follows. The potential energy in our model is the two-body operator (in units of b)

$$U = -\frac{2R}{L} \sum_{p,q,r} a^{\dagger}_{p+q} b^{\dagger}_{r-q} b_r a_p; \tag{7.3}$$

its expectation value can be calculated in terms of $D(q)$ (chapter 2). From the definition of $D(q)$, equation (2.94), we have

$$\langle U \rangle = -\frac{2R\beta_d}{L} \sum_{q} D(q); \tag{7.4}$$

In the case that we are studying, $\beta_d = n = 4$. The infinite sum has been calculated approximately by adding a finite number of points at small momentum transfer q, and beyond that the asymptotic form of $D(q)$ was used which gave a series that was summed analytically. The results are in table 7-1.

The rate at which $D(q)$ decreases remains almost the same for different values of the parameter L. For instance, both in figure 7-5 and in figure 7-6 ($L = 2R$ and $L = 50R$) $D(q)$ is close to 3×10^{-2} at $q = 4/R$. However, the density of points in the domain of $D(q)$ in a given interval is 25 times greater for $L = 50R$ than for $L = 2R$; therefore, the sum in equation (7.4) does not change drastically with L (the binding energy of the clusters amounts for most of the potential energy). For example compare the different values of $\langle U \rangle$ in table 7-1.

Table 7-1 also shows the total energy. We saw in chapter 5 that the eigenstates of the

Hamiltonian are easily obtained in terms of the Bethe momentum K; therefore, transforming equation (4.110) into units of b and comparing with equations (7.2) (and 7.4) we obtain the relation

$$(Re\ K)^2 - (Im\ K)^2 = \sum_p p^2 N(p) - \frac{L}{2\pi^2 R} \sum_q D(q). \tag{7.5}$$

The last column in table 7-1 are the results obtained from the Bethe momentum K (table 4-1), using the relation

$$\langle H \rangle = 4 \left(\frac{2\pi R}{L} \right)^2 \left[(Re\ K)^2 - (Im\ K)^2 \right]. \tag{7.6}$$

Equation (7.5) can be used to test the numerical results that we have obtained. If the numerical results for the quark-momentum distribution and the different-color correlation are correct, they must satisfy the sum rule defined by equation (7.5). The good agreement of the last two columns in table 7-1 (keeping in mind that both results include numerical error), indicates that this is in fact the case.

7.3 Elastic form factor

The exact results for the elastic form factor show that exchange terms are important (figures 7-7 and 7-8). At large momentum transfer q, both direct and exchange terms go to zero as q^{-6}; however, for small values of L ($L < L_o$, where $L_o \approx 8\pi$), the exchange term is greater than the direct one and the elastic form factor becomes negative (the points where F is negative are indicated by a double circle in figures 7-7 and 7-8). The larger the parameter L, the faster $F(q)$ decays.

The results derived from the impulse and cluster approximations are very different from the exact values. Their asymptotic form is q^{-4} instead of q^{-6}. The exchange term in the cluster approximation does not change the result from the impulse approximation significantly; therefore, the disagreement between the "theoretical" result and the "experimental" data is not due to quark exchange effects. The main source of the disagreement is the assumption of separability of the state of the deuteron, as the product of a term depending on cluster coordinates and another part that depends on internal coordinates of the nucleon.

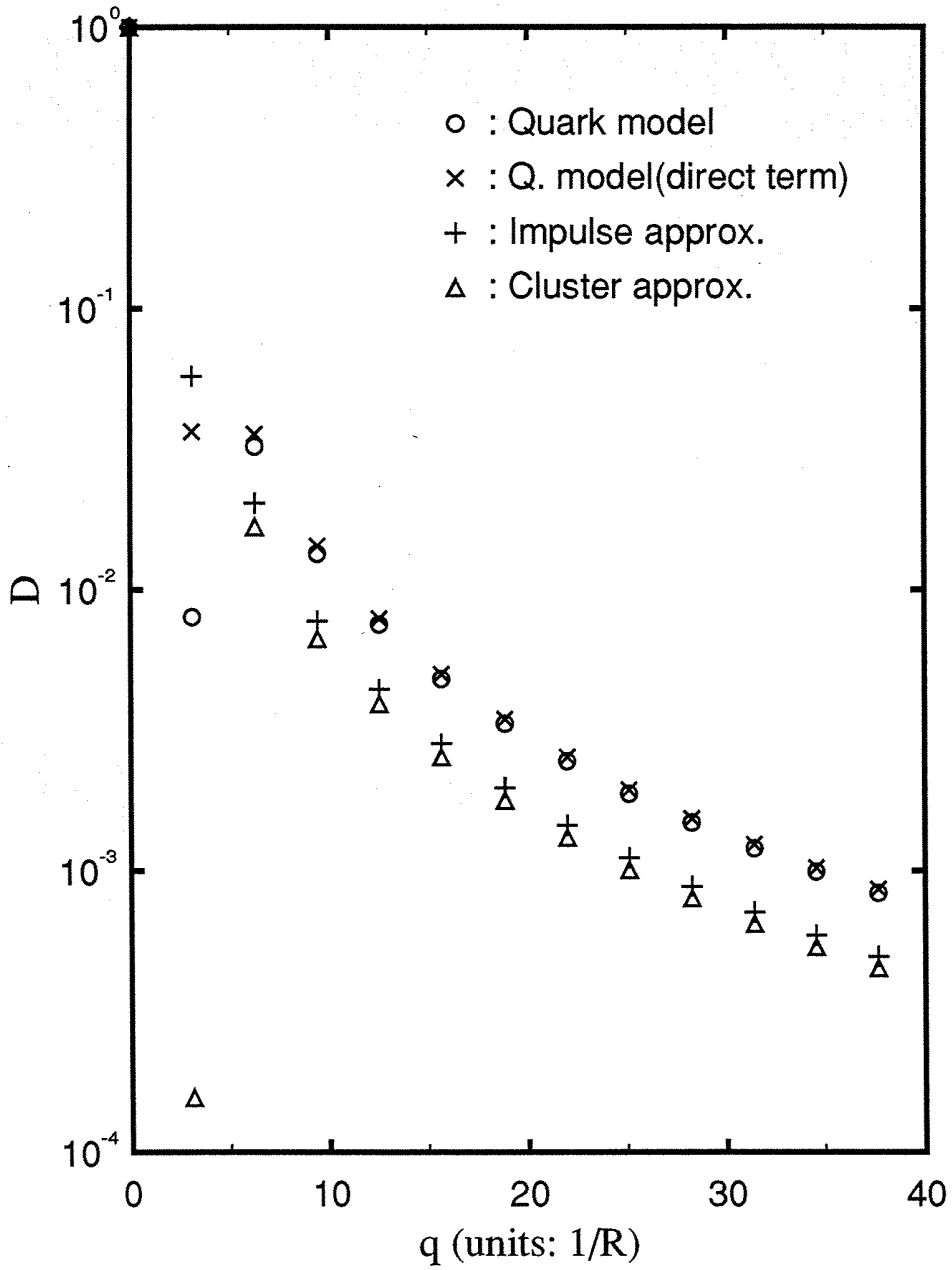


Figure 7-5: Different-color correlation function for $L = 2R$.

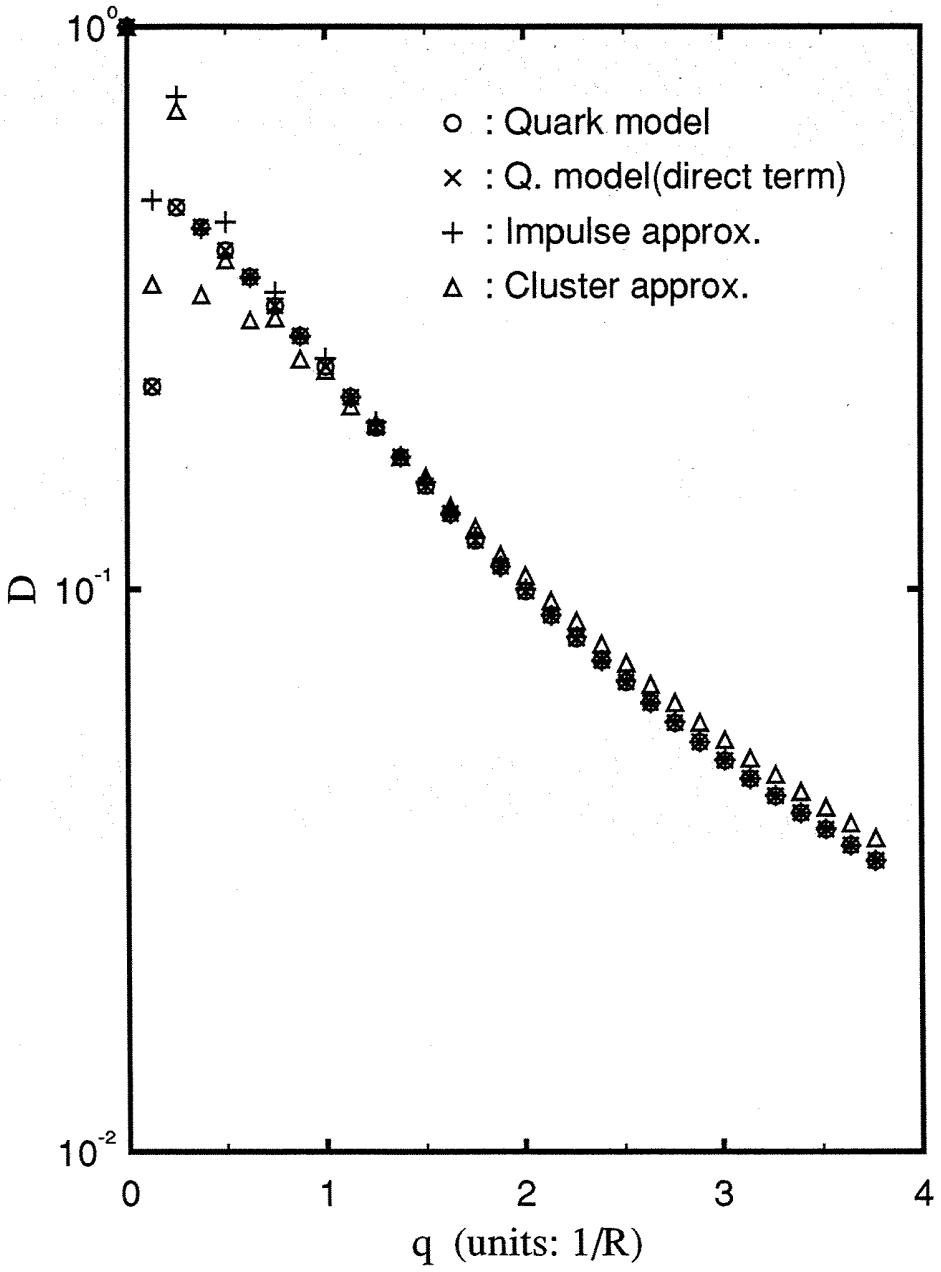


Figure 7-6: Different–color correlation function for $L = 50R$.

In the case $L = 5R$, due to the discrete nature of the momentum transfer, our model does not give any information between $q = 0$ and $q \approx 5/R$ ($\approx 1GeV$). At $q = 5/R$ the two approximations are close to the exact result, but for $q > 10/R$ they differ by several orders of magnitude.

7.4 Same–color correlation function

The same–color correlation function $C(q)$ gives the coefficients in the Fourier series for the probability of finding two quarks at some distance of each other:

$$P_c(x) = \frac{1}{L}\sum_q e^{i\left(\frac{2\pi}{L}\right)qx} C(q); \tag{7.7}$$

thus the probability that two quarks of the same color be at the same point is

$$P_c(0) = \frac{1}{L}\sum_q C(q). \tag{7.8}$$

But according to the exclusion principle this quantity must vanish and so must the sum of $C(q)$

$$\sum_q C(q) = 0. \tag{7.9}$$

The direct term of $C(q)$ is always positive and therefore it violates the exclusion principle; but when exchange is included $\sum_q C(q)$ becomes zero. Unlike the different–color correlation function, $C(q)$ decays faster the larger the value of L. The result obtained from the impulse approximation does not lead to $\sum_q C(q) = 0$, because it does not include quark exchange. Moreover, the domain of C_{IA} includes only every other term in the domain of $C(q)$ from the the quark model, as has been mentioned in chapter 6.

The cluster approximation introduces exchange thereby making the sum of $C(q)$ vanish. However, the function obtained is not a good approximation to the exact result. Even though C_{CA} has the correct domain, at every other point it is derived from the exchange term alone and in the other points from both direct plus exchange terms. As a result, C_{CA} fluctuates from point to point. The data for $L = 5R$ is shown in figure 7-9.

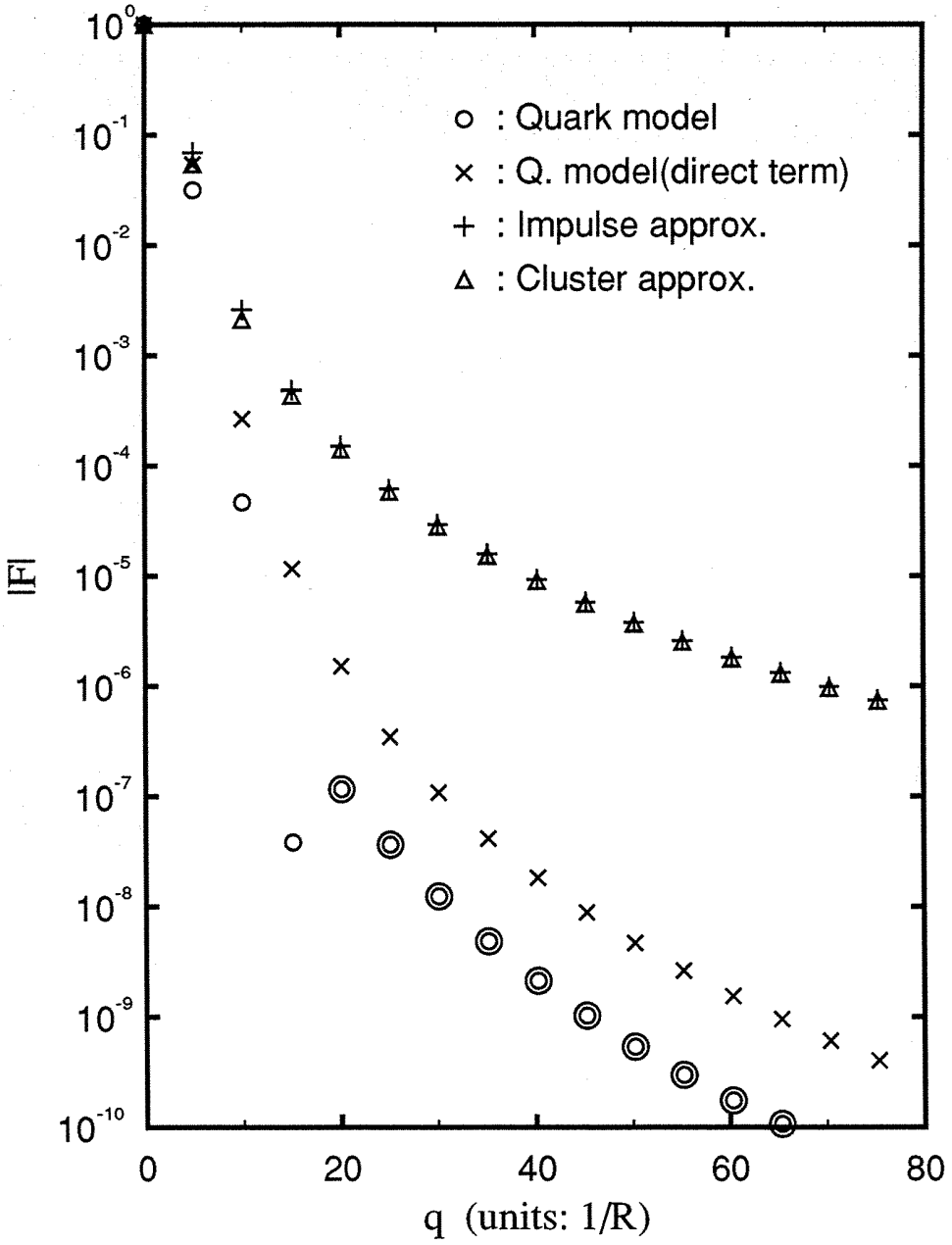


Figure 7-7: Elastic form factor for $L = 5R$. The double circles represent the points where the quark model result becomes negative.

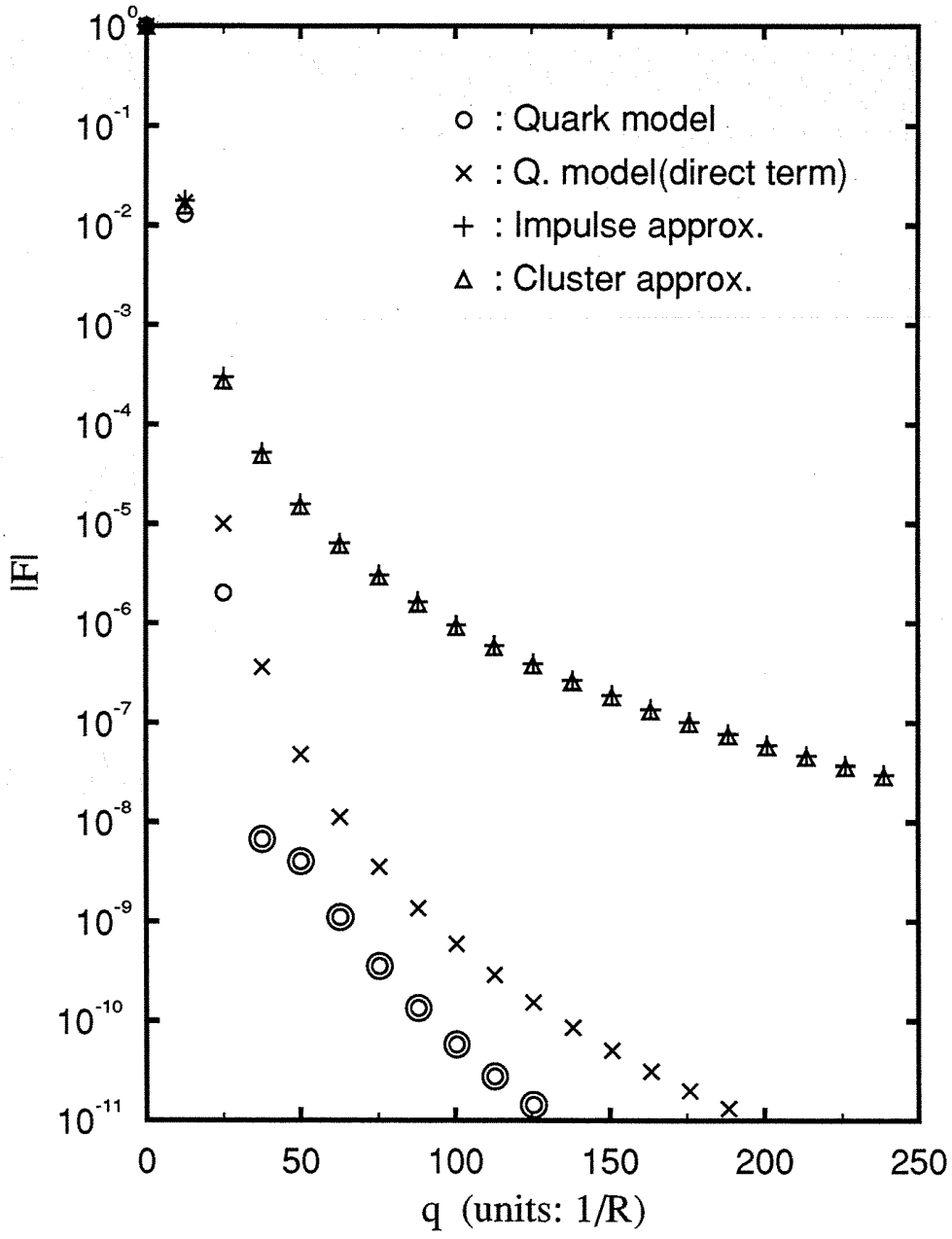


Figure 7-8: Elastic form factor for the high density case, $L = 2R$. The double circles represent the points where the quark model result becomes negative.

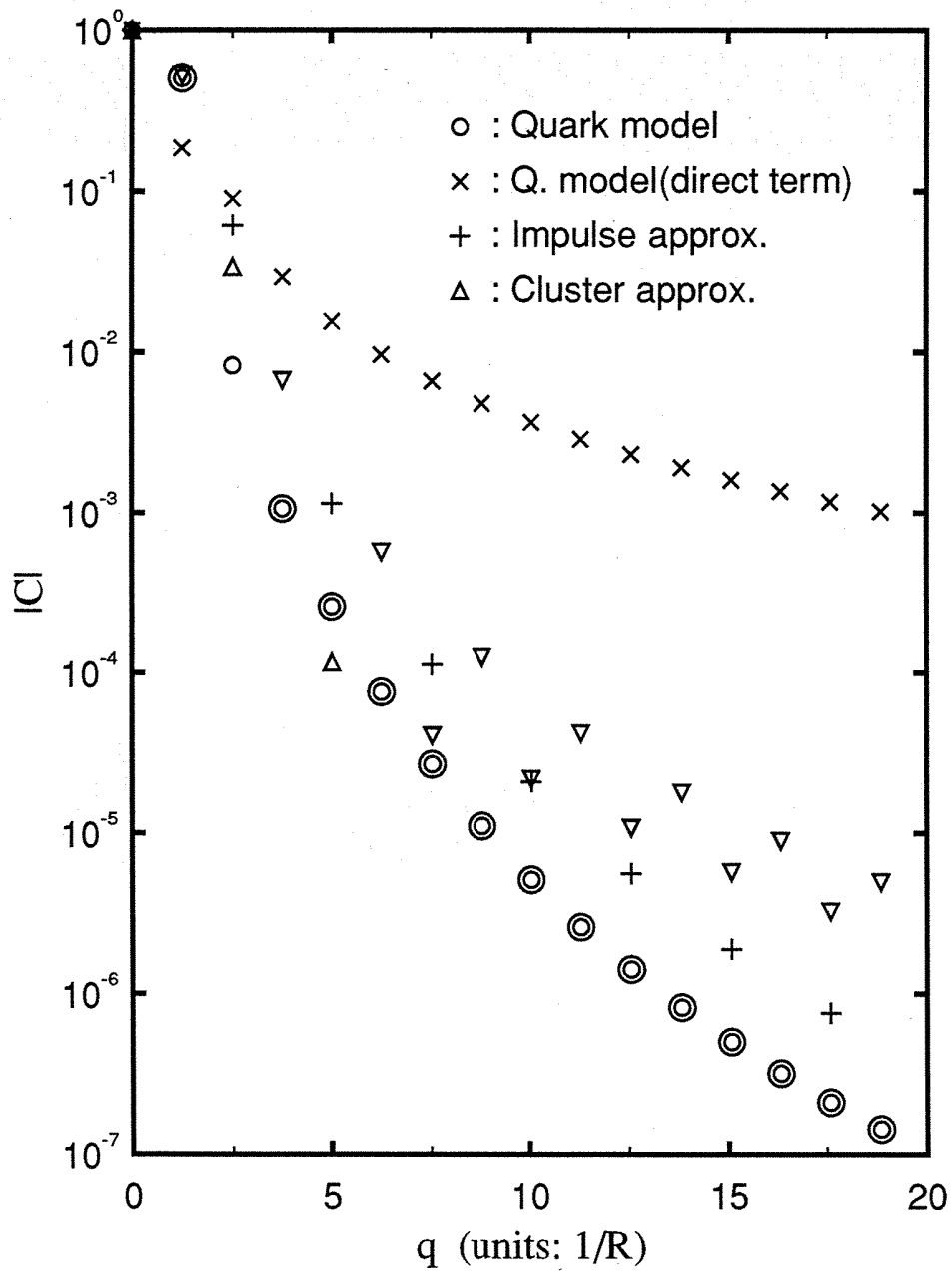


Figure 7-9: Same–color correlation function for $L = 5R$. The upside-down triangles and double circles represent those points where C is negative

7.5 Coulomb sum rule

The Coulomb sum rule can be calculated from the correlation functions and the elastic form factor. Equation (3.36) relates $\Re(q)$ to the total correlation function $P(q)$; comparing equations (2.90) and (2.92) we obtain the correlation function $P(q)$ in terms of the same- and different-color correlation functions, and thus the Coulomb sum rule becomes

$$\Re(q) = \frac{1}{4}[1 + C(q) + 2D(q)] - \mid F(q) \mid^2 . \tag{7.10}$$

The Coulomb sum rule $\Re(q)$ in the case of the deuteron saturates at about $5/R$ ($\approx 1GeV$), where it approaches the value $1/4$ (four quarks). For larger parameters L, it saturates even faster (see figures 7-10 and 7-11). In the observables discussed so far, the cluster approximation has always led to a direct term that agrees with the impulse approximation result, plus a small exchange correction. In the case of the Coulomb sum rule the situation would still be the same if we assumed that the nucleons were composite objects with certain internal spectrum that could be excited. This is not the way the impulse approximation is usually used; for example, in quasielastic scattering the nucleons are normally regarded as immutable objects whose internal stucture is fixed ("closure" approximation). With this assumption, the square of the elastic form factor of the nucleon can be factored out; therefore the Coulomb sum rule in the impulse approximation goes asymptotically to zero. On the other hand, in the cluster approximation by replacing the nucleon operators $A^\dagger$ by a superposition of quark operators, it is implicitly assumed that the complete space of the system is spanned by superpositions of four quark operators, which is consistent with the quark model.

The cluster approximation gives good results for the Coulomb sum rule (even if only the direct term is considered), while the impulse approximation does not. R_{IA} falls quickly to zero, in the case $L = 5R$; at larger volume L, it rises a little closer to the exact result (figures 7-10 and 7-11), because under those conditions the system looks more like two clusters without overlap.

Figure 7-12 shows the function $S(q)$ introduced in chapter 3:

$$S(q) \equiv \frac{2\,\Re(q)}{f^2(q)}, \tag{7.11}$$

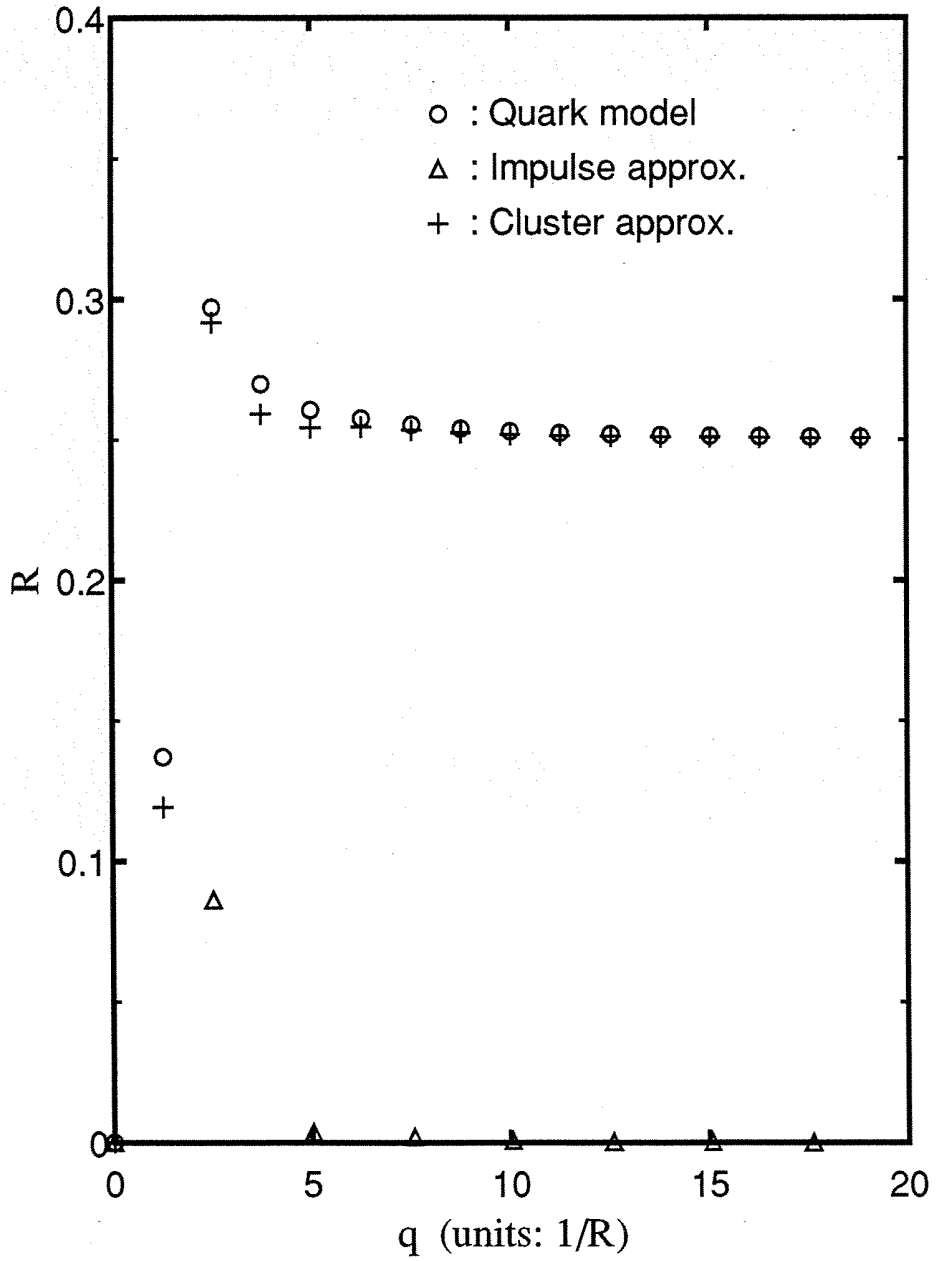


Figure 7-10: Coulomb sum rule for $L = 5R$.

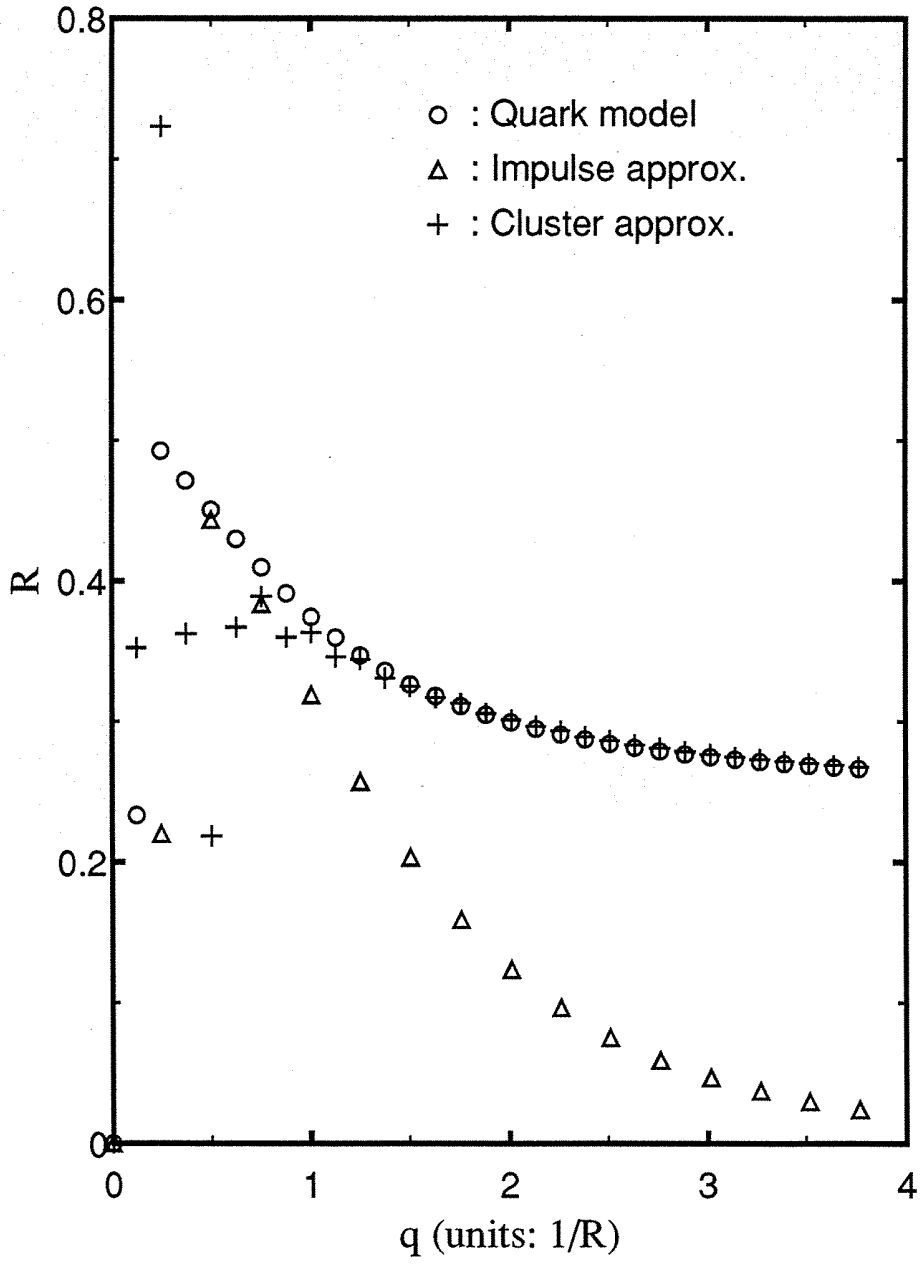


Figure 7-11: Coulomb sum rule for $L = 50R$.

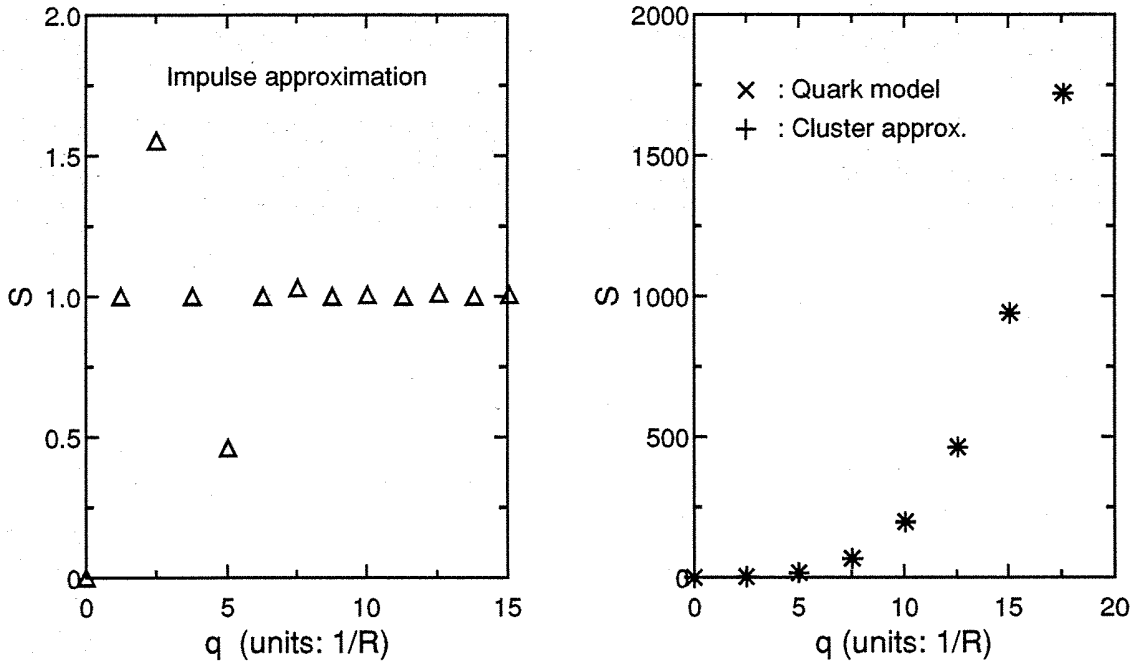


Figure 7-12: Hadronic Coulomb sum rule, $S(q)$, for $L = 5R$. On the left are the results from the impulse plus closure approximations, and on the right from the quark model and cluster approximation.

where $f(q)$ is the nucleon's elastic form factor. In the impulse approximation this function goes asymptotically to one. If one tries to define the same function in the quark model or cluster approximations (figure 7-12), which are both quark theories, the result would be a function that increases without limit, as pointed out by Horowitz [13].

In scattering experiments, both functions $\Re(q)$ and $S(q)$ are of interest. At small energy and momentum transfer, where the internal structure of the nucleons is not excited, $S(q)$ can be measured and it can be confirmed whether the target is made up by nucleons with a fixed form factor $f(q)$. Likewise $\Re(q)$ should prove to be of interest as we move into larger momentum and energy transfer.

Chapter 8

Summary and conclusions

The purpose of this project has been to test what kind of observables, related to scattering experiments, would be most sensitive to quark effects in nuclear targets. Our approach to the problem consisted in introducing a simple quark model of the nucleus, with the minimum aspects of quark dynamics that may lead to the emergence of quark effects. An effective hadron theory was extracted from the model, in which the nucleus is viewed as a system of nucleons, rather than quarks. The results from the quark model were considered as "experimental" data, and the search for quark effects was conducted by comparing those data with the "theoretical" predictions of the effective theory.

The quark model developed here has been based on the model for nuclear matter introduced by Koltun *et al*[17,18,19,20]. It has the advantage of leading to an exactly solvable Hamiltonian, with eigenstates of the Bethe *ansatz* type. In the ground state, the quarks are all bound into clusters; there is also a sequence of excited states which are of the cluster kind. This property has led to the existence of an effective hadron theory which reproduces the cluster energy spectrum very accurately. The effective Hamiltonian is also of the Bethe *ansatz* type and therefore exactly solvable. Another interesting feature of the model is that the cluster energy spectrum resembles that of a system of fermions, even in the case when the clusters contain only two quarks. In view of this property, we have restricted our treatment to quarks with only two, rather than three colors, which simplified the calculations considerably.

If the only role of quark dynamics in the nucleus were to fix the internal structure of the

individual nucleons, then the impulse approximation would be valid. To test this hypothesis, we have constructed an impulse approximation which has been used to calculate some dynamical functions. This calculations are done assuming two independent levels: the motion of nucleons inside the nucleus which is studied with the effective theory, and the motion of quarks inside the nucleon, described by the quark Hamiltonian. The intrinsic properties of a nucleon are considered to be independent of its interaction with other nucleons.

One of the reasons why the impulse approximation may fail is because of quark exchange. The wave function must be antisymmetric under the exchange of any two quarks, even if they belong to different nucleons. The effects of quark exchange have been obtained by introducing a cluster approximation. In this approximation scheme the nuclear states were derived from the eigenstates of the effective Hamiltonian, replacing the elementary nucleon creation operator by the operator that creates a cluster of quarks. The results from the cluster approximation have been separated into direct plus exchange terms. The direct terms lead to the same results of the impulse approximation, and the exchange terms are corrections due to quark exchange.

We studied a deuteron-like system, and investigated the behavior of several dynamical functions which are all related to scattering theory: the quark-momentum distribution, the quark-correlation functions, the elastic form factor, and the Coulomb sum rule. Analytical expressions were obtained for those observables, using an original technique which involves the use of two functions that we have dubbed ξ and S (appendices A and B). In developing this technique, two crucial elements have been the use of the momentum representation of the Bethe *ansatz*[22], and the simple form in which we rearranged the states of two clusters (appendix C).

Some selected results have been presented and discussed. These numerical results were obtained using the algorithms developed in appendices A and B. The results have been shown in units defined by two constants R and b, that are directly related to the diameter and mass of the nucleon. The parameter that determines the mathematical form of the observables is the ratio of the nuclear volume L, to the nucleon's diameter R. We have shown results for the cases $L = 2R$ and $L = 5R$ which is the range where the deuteron is.

Our results show that the two approximations based on the effective hadron theory are very successful in explaining the quark-momentum distribution and the different-color correlation function. This is not a coincidence, because these two observables are the ones that determine the value of the total energy; and in the construction of the effective theory, agreement with the energy spectrum was the guideline. Both approximations have failed to reproduce the asymptotic behavior of the elastic form factor. Furthermore, if the nucleons in the effective theory are considered as composite systems with an internal structure that can be excited, the resulting Coulomb sum rule is in good agreement with the quark model. However, if the intrinsic properties of the nucleon are considered invariant ("closure" approximation), the Coulomb sum rule becomes very different from the "experimental" data.

Contrary to what was expected, the results of the impulse approximation appear to be better than those of the cluster approximation. This striking result means that quark exchange is not the dominant source of error in the effective theory. The discrepancy with the "experimental" data arises mainly from the fact that the state of the deuteron cannot be factored into a part that depends only on cluster coordinates, and another part depending on the internal structure of the clusters. Therefore, we conclude that the anomalies found in the large momentum behavior of the elastic form factor and of the Coulomb sum rule are the signature of a quark dynamical effect, different from quark exchange. Based on the results of the model, we conjecture that in a real scattering experiment, the elastic form factor and the Coulomb sum rule may be the two observables which are the most sensitive to quark effects.

Our model does not lead to the property of the quark-momentum distribution implied by the EMC effect. Namely, our results for the deuteron are very close to the quark-momentum distribution of the isolated nucleon. We point out that the EMC effect is within a highly relativistic domain which escapes the scope of our model. Furthermore, the EMC effect is small in light targets such as the deuteron.

The hypothesis of nucleon swelling mentioned in the introduction does not account for the effect that we observed in the elastic form factor. If we assumed that the nucleon radius R were larger inside the nucleus, we would obtain a form factor closer to the "experimental" data. However, even if R is increased, the asymptotic behavior of $F(q)$ is still q^{-4}. The

"experimental" data decays asymptotically two orders of magnitude faster than that ($F(q) \sim q^{-6}$). In order to reproduce the exact behavior, we would have to assume a different functional shape for the form factor of the nucleon inside the nucleus.

To close this chapter, we will discuss some possible extensions of our model, and future projects inspired by it. First, it would be of interest to extend the model to the three-color case. We do not know the momentum representation of the Bethe *ansatz* in the three-color case, but we expect that it could be obtained as a generalization of the two-color problem, which is the case in configuration space. The analytic expressions for the observables could then be calculated in a similar way as done in the two-color case. We anticipate that we would encounter also functions S_4 and S_5, in addition to S_2 and S_3. The generalization of the functions S would be a simple task, following the technique of appendix B. However, even if the results can still be written in a compact form, the computing time necessary to obtain numerical results would increase significantly. The number of iterations in the algorithms increases as $n!^2$, as explained at the end of chapter 6. This complication does not completely rule out the possibility of studying the three-color problem, since in the two-color case the computing times are only of the order of a few minutes of CPU in a VAX 8600 computer.

Another extension of our model that also seems feasible and of interest is the study of systems with a larger number of clusters. If the number of clusters is increased, we still have the advantage that the eigenstates are known [22]; however, writing those states in a simple form as we did in the case of two clusters may not be an easy task. Furthermore, we have again the same complication as in introducing more colors; i. e. the length of the algorithms increases very rapidly as the number of quarks increases. More degrees of freedom could be introduced to make the model more realistic; for example spin and isospin could be included. Kebukawa[45] has generalized his result in momentum representation to the case of spin and isospin 1/2; however, this extension makes the problem of calculating dynamical functions very difficult.

Personally, we find more attractive a further investigation of the anomalies encountered in the elastic form factor and Coulomb sum rule, than an extension of the model. The very different behavior of the Coulomb sum rule when the "closure" approximation is introduced,

can be used to test whether the internal structure of clusters plays an important role within the quasielastic domain. The following *experiment* could provide such a test: Suppose we only look at scattering events in which the energy transfer has an upper bound; the bound, ω_{max} could be chosen such that the energy transfer is not enough to excite the clusters from their internal ground state. The response function $R(q,\omega)$ can then be calculated in the quark model, and under the condition of an upper bound in ω, the Coulomb sum rule becomes

$$\Re_e(q) \equiv \lim_{\epsilon \to 0^+} \sum_{\omega=\frac{q^2}{2M}+\epsilon}^{\omega_{max}} \Re(q,\omega); \tag{8.1}$$

and from it, the function $S_e(q)$ is obtained

$$S_e(q) \equiv n\frac{\Re_e(q)}{f^2(q)}. \tag{8.2}$$

We believe that there are two possible outcomes of the experiment. First, it may happen that $S_e(q)$ goes asymptotically to one, as it should be the case if the form factor of the nucleons is in fact a constant and can be factored in the function $\Re_e(q)$. But, as we found in the case of the elastic form factor, if one insists in separating it as the product of a hadronic part times the intrinsic form factor of the nucleon, the last one has to be modified. Assuming a bigger effective radius R would not be enough. A different functional form $\widetilde{f}(q)$ would have to be introduced, which should decay faster than $f(q)$. We conjecture that the same may happen when one tries to write $\Re_e(q)$ as a product of a nucleonic correlation function times the square of the nucleon's form factor; the nucleon's form factor $f(q)$ may have to be replaced by $\widetilde{f}(q)$. Thus, the denominator in 8.2 would decay slower than $\Re_e$ and the result would go asymptotically to zero.

In order to conduct the proposed experiment, we must be able to calculate the response function. This seems feasible, since the inelastic form factor can be computed in a similar way as the elastic one; then the problem would be to identify which final states contribute to the response function. Since the energy transfer has been given an upper bound, that should be a tractable problem. It remains as a topic for future research.

Appendix A

Analytical evaluation of series

All of the infinite sums that we obtain in the quark model and the approximation schemes can be summed analytically, because the functions being summed are simple rational functions. If $f(z)$ is an analytic function on the complex plane, with only isolated singularities, the following relations are valid as long as the series are convergent [46, page 175] .

$$\sum_n f(n) = -\sum_{res.f} Res\left[\pi f(z)\cot(\pi z)\right], \tag{A.1}$$

$$\sum_{n+1/2} f(n) = \sum_{res.f} Res\left[\pi f(z)\tan(\pi z)\right], \tag{A.2}$$

where the sums on the right hand sides run over the singularities of the function $f(z)$, and *Res* stands for the residue of the function inside the square brackets at the points where the sum is evaluated.

We will introduce an original algorithm to calculate the series, in the case when $f(z)$ is a rational function with numerator equals to one. In that case a convenient way of finding the residues of $f(z)\cot(\pi z)$ consists on expanding $f(z)$ in partial fractions. But even if the denominator of $f(z)$ is a polynomial of degree as low as ten, the partial fraction expansion becomes cumbersome, and even with the help of the computer, we still have the problem that its algebraic form may take several pages. We need a more compact way of writing the result; instead of attempting to write down all the partial fractions in the expansion, we will explain how they can be derived iteratively. To that purpose we will define a couple of functions ξ and η.

Definition 1 *If $\{z_1, z_2, \ldots, z_n\}$ is a set of n complex numbers, none of which is an integer, the function ξ of order n is defined in the following way*

$$\xi_n(z_1, z_2, \ldots, z_n) \equiv \sum_m (m - z_1)^{-1}(m - z_2)^{-1} \cdots (m - z_n)^{-1}. \tag{A.3}$$

Definition 2 *If $\{z_1, z_2, \ldots, z_n\}$ is a set of n complex numbers, none of which is an integer*

$$\eta_n(z_1, z_2, \ldots, z_n) \equiv \sum_{m+1/2} (m - z_1)^{-1}(m - z_2)^{-1} \cdots (m - z_n)^{-1}. \tag{A.4}$$

The series that define ξ_n and η_n are convergent for any $n = 1, 2, \ldots$, and *absolutely convergent* if n is greater than 1; ξ and η are related by the relation

$$\eta_n(z_1, z_2, \ldots, z_n) = \xi_n(z_1 + 1/2, z_2 + 1/2, \ldots, z_n + 1/2); \tag{A.5}$$

in the remainder of this appendix we will concentrate on the function ξ only and most of our results will be also valid for η, or can be easily generalized. The order of the arguments of ξ is irrelevant:

Proposition 4 *If $\nu_1, \nu_2, \ldots, \nu_n$ is any permutation of order n,*

$$\xi_n(z_1, z_2, \ldots, z_n) = \xi_n(z_{\nu_1}, z_{\nu_2}, \ldots, z_{\nu_n}). \tag{A.6}$$

When the arguments of ξ are repeated, we will write down only one of them, if it is clear how they must be repeated; for example $\xi_n(z)$ stands for

$$\xi_n(z) \equiv \xi_n(z, z, \ldots, z); \tag{A.7}$$

in this case it is clear that the argument z appears n times, because the index in ξ_n tells us so. Another example of short notation is

$$\xi_4(a \pm b, c \pm d) \equiv \xi_4(a + b, a - b, c + d, c - d); \tag{A.8}$$

the subindex of the function ξ will be dropped if it is one:

$$\xi(z) \equiv \xi_1(z). \tag{A.9}$$

The function ξ_n is completely defined analytically, for any n greater or equal to 1, by the following three properties.

Proposition 5

$$\xi_1(z) = -\pi \cot(\pi z). \tag{A.10}$$

Proposition 6 *If z_n is different from z_{n-1}, then*

$$\xi_n(z_1, \ldots, z_{n-1}, z_n) = \frac{\xi_{n-1}(z_1, \ldots, z_{n-2}, z_{n-1}) - \xi_{n-1}(z_1, \ldots, z_{n-2}, z_n)}{z_{n-1} - z_n}. \tag{A.11}$$

Proposition 7 *If the n arguments of the function ξ_n are all equal*

$$\xi_n(z) = \sum_{k=\epsilon,\epsilon+2,\ldots,n-\epsilon} \left\{ {n \atop k} \right\} \frac{\pi^{n-k}}{(n-1)!} \left[\xi_1(z)\right]^k, \tag{A.12}$$

where the number denoted by the curly braces is defined iteratively by

$$\left\{ {n \atop k} \right\} = (k-1) \left\{ {n-1 \atop k-1} \right\} + (k+1) \left\{ {n-1 \atop k+1} \right\}, \tag{A.13}$$

$$\left\{ {1 \atop 0} \right\} \equiv \left\{ {1 \atop 1} \right\} \equiv 1. \tag{A.14}$$

Proposition 8 *The second order function ξ_2 has the form*

$$\xi_2(z_1, z_2) = \frac{\pi^2 \, j_o\left[\pi(z_1 - z_2)\right]}{\sin(\pi z_1) \sin(\pi z_2)}, \tag{A.15}$$

where j_o is the spherical Bessel function

$$j_o(z) = \frac{\sin z}{z}. \tag{A.16}$$

In what follows we will list some properties of the function ξ which are used throughout this dissertation.

Proposition 9

$$\xi_n(z_1 + m, z_2 + m, \ldots, z_n + m) = \xi_n(z_1, z_2, \ldots, z_n) \quad \forall m \in \mathbf{Z}. \tag{A.17}$$

Proposition 10

$$\xi_n(z_1^\star, z_2^\star, \ldots, z_n^\star) = \xi_n^\star(z_1, z_2, \ldots, z_n). \tag{A.18}$$

Proposition 11

$$\xi_n(-z_1,-z_2,\ldots,-z_n) = (-1)^n\,\xi_n(z_1,z_2,\ldots,z_n). \tag{A.19}$$

Proposition 12

$$\xi_2(z+m,z) = \delta_{m,0}\,\xi_2(z). \tag{A.20}$$

Proposition 13

$$\xi_n(z+m,z,\ldots,z) = \delta_{m,0}\,\xi_n(z). \tag{A.21}$$

Proposition 14

$$\xi_n(ix) = \begin{cases} \textit{real}, & \textit{if } n \textit{ is even} \\ \textit{purely imaginary}, & \textit{if } n \textit{ is odd} \end{cases}, \forall x \in \mathbf{R} \tag{A.22}$$

Proposition 15

$$\xi_n(z_1,\ldots,z_n) = 2^{-n}\left[\xi_n(z_1/2,\ldots,z_n/2) + \eta_n(z_1/2,\ldots,z_n/2)\right]. \tag{A.23}$$

Proposition 16 *If $z_i \neq z_{i+m}$ for $i = 1,2,\ldots,m$, then*

$$\xi_{2m}(z_1,\ldots,z_m,z_{m+1},\ldots,z_{2m}) = \left[\prod_{i=1}^{m}(z_i - z_{i+m})^{-1}\right] \tag{A.24}$$

$$\times \sum_{l_1,\ldots,l_m=0}^{1} (-1)^{l_1+\cdots+l_m}\xi_m(z_{1+l_1m}, z_{2+l_2m},\ldots,z_{m+l_mm}). \tag{A.25}$$

This last property is very useful in reducing the order of ξ when one wants to write down the explicit algebraic form of it. When the arguments of ξ_{2m} are only two, each one repeated m times, this property takes the simpler form:

Proposition 17 *If $z_1 \neq z_2$, then*

$$\xi_{2m}(z_1,z_2) = (z_1-z_2)^{-m}\sum_{l=0}^{m}(-1)^l\binom{m}{l}\xi_m(\overbrace{z_1,\ldots,z_1}^{l},\overbrace{z_2,\ldots,z_2}^{m-l}). \tag{A.26}$$

Proposition 18 *If the complex numbers $v_1, v_2, \ldots, v_m$ are all different from zero, then*

$$\xi_{2m}(u_1 \pm v_1,\ldots,u_m \pm v_m) = \frac{2^{-m}}{v_1v_2\cdots v_m}\sum_{l_1,\ldots,l_m=0}^{1}(-1)^{l_1+\cdots+l_m}\xi_m(u_1+v_1^{l_1},\ldots,u_m+v_m^{l_m}), \tag{A.27}$$

where the quantities v_m^l *are defined as*

$$v_m^l \equiv (-1)^l \, v_m \tag{A.28}$$

Proposition 19 *If* p *and* z *are different from zero, then*

$$\xi_4(p \pm z, \pm z) = \left(\frac{2}{z}\right) \frac{\xi(z)}{p^2 - 4z^2} = -\left(\frac{2\pi}{z}\right) \frac{\cot(\pi z)}{p^2 - 4z^2}. \tag{A.29}$$

In the next chapter we will calculate some multiple series using some of the properties given here. We have not included a proof of the propositions, since most of them are straight forward.

Appendix B

Evaluation of multiple series

In this appendix we investigate some multiple series which are encountered throughout the dissertation. The sum over each of the indices is evaluated using the technique described in appendix A; the absolute convergence of the series defined there imply that the sums can be computed in any order.

B.1 Three simple multiple series

We will start by obtaining the analytic form of three simple multiple series which are found in our model. We will call them S_1, S_2 and S_3.

Definition 3 *If n is a positive integer and $\{z_1, z_2, \ldots, z_n\}$ are complex numbers, none of them integer, then the function $S_1(z_1, \ldots, z_n)$ is defined as*

$$S_1(z_1, \ldots, z_n) \equiv \sum_{p_1, \ldots, p_n} \delta(p_1 + \ldots + p_n)\,(p_1 - z_1)^{-1}(p_2 - z_2)^{-1} \cdots (p_n - z_n)^{-1}. \qquad \text{(B.1)}$$

Proposition 20

$$S_1(z_1, \ldots, z_n) = \frac{(-\pi)^n\, j_0\left[\pi(z_1 + \cdots + z_n)\right]}{\sin(\pi z_1) \cdots \sin(\pi z_n)}, \qquad \text{(B.2)}$$

where j_0 is the spherical Bessel function of order 0.

Proof. We first prove it for the simple case $n = 1$. Since z_1 is not an integer, $\sin(\pi z_1)$ is different from zero and we have

$$\begin{aligned} S_1(z_1) &= \sum_{p_1} \frac{\delta(p_1)}{p_1 - z_1} = -\frac{1}{z_1} \\ &= -\frac{\sin(\pi z_1)}{z_1 \sin(\pi z_1)} = -\frac{\pi\, j_0(\pi z_1)}{\sin(\pi z_1)}. \end{aligned}$$

Now suppose that the proposition is valid for an arbitrary n.

$$\sum_{p_1,\ldots,p_n} \delta(p_1 + \ldots + p_n)\,(p_1 - z_1)^{-1} \cdots (p_n - z_n)^{-1} = \frac{(-\pi)^n\, j_0\,[\pi(z_1 + \cdots + z_n)]}{\sin(\pi z_1) \cdots \sin(\pi z_n)};$$

if z_1 is replaced by $z_1 + p_{n+1}$, where p_{n+1} is an integer, we have

$$\sum_{p_1,\ldots,p_n} \delta(p_1 + \ldots + p_n)(p_1 - z_1 - p_{n+1})^{-1}(p_2 - z_2)^{-1} \cdots (p_n - z_n)^{-1} = \frac{(-\pi)^n\, j_0\,[\pi(z_1 + \cdots + z_n + p_{n+1})]}{\sin \pi(z_1 + p_{n+1}) \cdots \sin(\pi z_n)}.$$

The dummy index p_1 on the left hand side can be replaced by $p_1 + p_{n+1}$

$$\sum_{p_1,\ldots,p_n} \delta(p_1 + \ldots + p_{n+1})\,(p_1 - z_1)^{-1} \cdots (p_n - z_n)^{-1} = \frac{(-\pi)^n\, j_0\,[\pi(z_1 + \cdots + z_n + p_{n+1})]}{(-1)^{p_{n+1}} \sin(\pi z_1) \cdots \sin(\pi z_n)};$$

multiplying both sides times $(p_{n+1} - z_{n+1})^{-1}$, and taking the sum over p_{n+1} leads to

$$S_1(z_1, \ldots, z_n, z_{n+1}) = \frac{(-\pi)^n}{\sin(\pi z_1) \cdots \sin(\pi z_n)} \sum_{p_{n+1}} \frac{j_0\,[\pi(z_1 + \cdots + z_n + p_{n+1})]}{(-1)^{p_{n+1}}(p_{n+1} - z_{n+1})}. \tag{B.3}$$

There are two possibilities. First, the sum $z_1 + \cdots + z_n$ can be an integer m, in which case we have

$$j_0\,[\pi(z_1 + \cdots + z_n + p_{n+1})] = \delta(p_{n+1} + m);$$

and equation (B.3) becomes

$$\begin{aligned} S_1(z_1, \ldots, z_n, z_{n+1}) &= -\frac{(-1)^m(-\pi)^n}{\sin(\pi z_1) \cdots \sin(\pi z_n)(m + z_{n+1})} \\ &= \frac{(-\pi)^{n+1}}{\sin(\pi z_1) \cdots \sin(\pi z_n) \sin(\pi z_{n+1})} \left[\frac{\sin\,[\pi(m + z_{n+1})]}{\pi(m + z_{n+1})}\right] \\ &= \frac{(-\pi)^{n+1}\, j_0\,[\pi(z_1 + \cdots + z_{n+1})]}{\sin(\pi z_1) \cdots \sin(\pi z_{n+1})}; \end{aligned} \tag{B.4}$$

which is what we want to prove.

The second possibility in equation (B.3) is that the sum $z_1 + \cdots + z_n$ is not an integer, in which case the function j_o can be written in terms of the sine function, and equation (B.3) becomes

$$S_1(z_1, \ldots, z_n, z_{n+1}) = \frac{(-\pi)^n \, \sin\left[\pi(z_1 + \cdots + z_n)\right]}{\pi \sin(\pi z_1) \cdots \sin(\pi z_n)} \sum_{p_{n+1}} (p_{n+1} + z_1 + \cdots + z_n)^{-1} (p_{n+1} - z_{n+1})^{-1};$$

the last sum can be easily evaluated using the results from appendix A

$$\begin{aligned} \sum_{p_{n+1}} (p_{n+1} + z_1 + \cdots + z_n)^{-1} (p_{n+1} - z_{n+1})^{-1} &= \xi_2(z_{n+1}, -z_1 - \cdots - z_n) \\ &= -\pi^2 \frac{j_0\left[\pi(z_1 + \cdots + z_{n+1})\right]}{\sin(\pi z_{n+1}) \sin(\pi(z_1 + \cdots + z_n))}; \end{aligned}$$

therefore, we have

$$S_1(z_1, \ldots, z_n, z_{n+1}) = \frac{(-\pi)^{n+1} \, j_0\left[\pi(z_1 + \cdots + z_{n+1})\right]}{\sin(\pi z_1) \cdots \sin(\pi z_{n+1})};$$

which concludes the proof.

The next function that we will define includes some denominators which may appear twice, making its evaluation more complicated.

Definition 4 *If $\{z_1, z_2, z_3, z_4\}$ are complex numbers, none of them integer, then*

$$S_2(z_1, z_2, z_3, z_4) \equiv \sum_{p_1, p_2} (p_1 - z_1)^{-1} (p_1 - z_2)^{-1} (p_2 - z_3)^{-1} (p_1 + p_2 - z_4)^{-1}. \tag{B.5}$$

If we evaluate the sum over p_2 we get the result

$$S_2(z_1, z_2, z_3, z_4) = \sum_{p} (p - z_1)^{-1} (p - z_2)^{-1} \, \xi_2(z_3, z_4 - p). \tag{B.6}$$

If $z_4 - z_3 = m$, where m is an integer; using proposition 11, from appendix A, we have

$$\xi_2(z_3, z_4 - p) = \delta_{p,m} \, \xi_2(z_3);$$

which leads to the following result.

Proposition 21 *If $z_4 - z_3 = m$, where m is an integer, then*

$$S_2(z_1, z_2, z_3, z_4) = \frac{\xi_2(z_3)}{(z_4 - z_3 - z_1)(z_4 - z_3 - z_2)}. \tag{B.7}$$

If z_3 and z_4 do not differ by an integer, then the two arguments of the function ξ in equation B.6 are never equal, and from proposition 6, appendix A, we have

$$\xi_2(z_3, z_4 - p) = \frac{\xi_1(z_3) - \xi_1(z_4)}{p + z_3 - z_4};$$

and the result of the sum over p in S_2 leads to

Proposition 22 *If $z_4 - z_3$ is not an integer,*

$$S_2(z_1, z_2, z_3, z_4) = [\xi_1(z_3) - \xi_1(z_4)]\ \xi_3(z_1, z_2, z_4 - z_3). \tag{B.8}$$

Definition 5 *If $\{z_1, \ldots, z_6\}$ are complex numbers, none of them integer, then*

$$\begin{aligned} S_3(z_1, z_2, z_3, z_4, z_5, z_6) \equiv & \sum_{p_1, p_2, p_3} (p_1 - z_1)^{-1}(p_1 - z_2)^{-1} \\ & \times (p_2 - z_3)^{-1}(p_2 - z_4)^{-1}(p_3 - z_5)^{-1}(p_1 + p_2 + p_3 - z_6)^{-1}. \end{aligned} \tag{B.9}$$

Once again, the order of the sums is arbitrary; evaluation of the sum over p_3 gives

$$S_3(z_1, \ldots, z_6) = \sum_{p_1, p_2} (p_1 - z_1)^{-1}(p_1 - z_2)^{-1}(p_2 - z_3)^{-1}(p_2 - z_4)^{-1}\ \xi_2(z_5, z_6 - p_1 - p_2). \tag{B.10}$$

If $z_6 - z_5 = m$, where m is an integer, property A.8 leads to

$$\xi_2(z_5, z_6 - p_1 - p_2) = \delta(p_1 + p_2 - m)\ \xi_2(z_5);$$

and the sum over p_2 becomes trivial

$$S_3(z_1, z_2, z_3, z_4, z_5, z_6) = \xi_2(z_5) \sum_p (p - z_1)^{-1}(p - z_2)^{-1}(m - p - z_3)^{-1}(m - p - z_4)^{-1}.$$

Proposition 23 *If $z_6 - z_5 = m$, where m is an integer,*

$$S_3(z_1, z_2, z_3, z_4, z_5, z_6) = \xi_2(z_5)\ \xi_4(z_1, z_2, z_6 - z_5 - z_3, z_6 - z_5 - z_4). \tag{B.11}$$

If z_5 and z_6 do not differ by an integer, from properties A.3 and A.5 we have

$$\xi_2(z_5, z_6 - p_1 - p_2) = \frac{\xi_1(z_5) - \xi_1(z_6)}{p_1 + p_2 + z_5 - z_6},$$

and the sum over p_2 in equation B.10 becomes a function ξ_3

$$S_3(z_1,z_2,z_3,z_4,z_5,z_6) = [\xi_1(z_5) - \xi_1(z_6)] \sum_p (p-z_1)^{-1}(p-z_2)^{-1}\xi_3(z_3,z_4,z_6-z_5-p). \quad \text{(B.12)}$$

When $z_6 - z_5 - z_3$, and $z_6 - z_5 - z_4$ are not integers, the use of property A.3 leads to

$$\begin{aligned}\xi_3(z_3,z_4,z_6-z_5-p) &= \frac{\xi_2(z_3,z_4) - \xi_2(z_3,z_6-z_5-p)}{p+z_4+z_5-z_6}, \\ &= \frac{\xi_2(z_3,z_4)}{p+z_4+z_5-z_6} + \frac{\xi_1(z_6-z_5)-\xi_1(z_3)}{(p+z_4+z_5-z_6)(p+z_3+z_5-z_6)}.\end{aligned}$$

Using this result in equation B.12 leads to the following.

Proposition 24 *If* $(z_6 - z_5)$, $(z_6 - z_5 - z_3)$, *and* $(z_6 - z_5 - z_4)$ *are not integers,*

$$\begin{aligned}S_3(z_1,z_2,z_3,z_4,z_5,z_6) &= [\xi_1(z_5) - \xi_1(z_6)] \\ &\times \{\xi_2(z_3,z_4)\,\xi_3(z_1,z_2,z_6-z_5-z_4) + [\xi_1(z_6-z_5) - \xi_1(z_3)]\,\xi_4(z_1,z_2,z_6-z_5-z_4,z_6-z_5-z_3)\}.\end{aligned}$$

When $(z_6 - z_5 - z_i)$ is not an integer but $(z_6 - z_5 - z_j)$ equals an integer m, where (i,j) are either (3,4) or (4,3), then

$$\xi_3(z_3,z_4,z_6-z_5-p) = \frac{\xi_2(z_3,z_4) - \delta_{p,m}\,\xi_2(z_j)}{p+z_i+z_5-z_6};$$

and from equation B.12 we get:

Proposition 25 *If* $(z_6 - z_5)$ *and* $(z_6 - z_5 - z_i)$ *are not integers but* $(z_6 - z_5 - z_j)$ *equals an integer, where* (i,j) *are either (3,4) or (4,3), then*

$$\begin{aligned}S_3(z_1,z_2,z_3,z_4,z_5,z_6) &= [\xi_1(z_5) - \xi_1(z_6)] \qquad \text{(B.13)} \\ &\times \left[\xi_2(z_3,z_4)\,\xi_3(z_1,z_2,z_6-z_5-z_i) - \frac{\xi_2(z_j)}{(z_6-z_5-z_j-z_1)(z_6-z_5-z_j-z_2)(z_i-z_j)}\right].\end{aligned}$$

If $z_6 - z_5 - z_3 = l$ and $z_6 - z_5 - z_4 = m$, with m and l two different integers

$$\xi_3(z_3,z_4,z_6-z_5-p) = \frac{\delta_{p,l}\,\xi_2(z_3) - \delta_{p,m}\,\xi_2(z_4)}{z_3-z_4};$$

which with equation B.12 becomes

Proposition 26 *If $(z_6 - z_5)$ is not an integer and $z_6 - z_5 - z_3 = l$, $z_6 - z_5 - z_4 = m$, where l and m are different integers,*

$$S_3(z_1, z_2, z_3, z_4, z_5, z_6) = \left(\frac{\xi_1(z_5) - \xi_1(z_6)}{z_3 - z_4}\right) \times \left[\frac{\xi_2(z_3)}{(z_6 - z_5 - z_3 - z_1)(z_6 - z_5 - z_3 - z_2)} - \frac{\xi_2(z_4)}{(z_6 - z_5 - z_4 - z_1)(z_6 - z_5 - z_4 - z_2)}\right]. \tag{B.14}$$

And finally we can have $z_6 - z_5 - z_3 = z_6 - z_5 - z_4 = m$, where m is an integer; in that case, from property A.9 we have

$$\xi_3(z_3, z_4, z_6 - z_5 - p) = \delta_{p,m}\xi_3(z_3).$$

Proposition 27 *If $z_5 - z_6$ is not an integer, but $z_6 - z_5 - z_3 = z_6 - z_5 - z_4 =$ integer, then*

$$S_3(z_1, z_2, z_3, z_4, z_5, z_6) = \frac{\xi_3(z_3)[\xi_1(z_5) - \xi_1(z_6)]}{(z_6 - z_5 - z_4 - z_1)(z_6 - z_5 - z_4 - z_2)}. \tag{B.15}$$

B.2 Momentum distribution in the cluster approximation

We will now calculate the exchange term of the quark momentum distribution, in the cluster approximation. We found in chapter 6 that

$$N_e^h(r) = -\frac{\beta^4\beta_h^2}{64}\sum_p \frac{X_8(p)}{(p^2 + \alpha^2)\left[(p + r)^2 - K^2\right]}, \tag{B.16}$$

with X_8 given by

$$X_8 \equiv \xi_8(\pm i\alpha, r \pm K, 2r + 2p \pm i\alpha, r + p/2 \pm i\alpha/2). \tag{B.17}$$

To evaluate such a sum we must first find the algebraic dependence of X_8 on the integer p, with the help of the results from appendix A; from property A.14 we have

$$X_8 = \frac{1}{8K(i\alpha)^3}\sum_{j,l,m,n=0}^{1}(-1)^{j+l+m+n}\xi_4(i\alpha_j, r + K_l, 2r + 2p + i\alpha_m, r + p/2 + i\alpha_n/2); \tag{B.18}$$

the subindices introduced for α and K stand for

$$\alpha_l \equiv (-1)^l\alpha, \qquad K_l \equiv (-1)^l K. \tag{B.19}$$

the terms on the sums in equation B.18 come in complex conjugate pairs because

$$\xi_4(i\alpha_o, r+K_l, 2r+2p+i\alpha_m, r+p/2+i\alpha_n/2) = \xi_4^*(i\alpha_1, r+K_l, 2r+2p+i\alpha_{1-m}, r+p/2+i\alpha_{1-n}/2); \tag{B.20}$$

hence, the sum over j gives twice the imaginary part of the term with $j = 0$

$$X_8 = -\frac{1}{4K\alpha^3} Im \left[\sum_{l,m,n=0}^{1} (-1)^{l+m+n} \xi_4(i\alpha, r+K_l, 2r+2p+i\alpha_m, r+p/2+i\alpha_n/2) \right]; \tag{B.21}$$

using property A.3 we get

$$X_8 = -\frac{1}{4K\alpha^3} Im \left[\sum_{j,l,m,n=0}^{1} (-1)^{j+l+m+n} c_l \xi_3(2r+2p+i\alpha_m, r+p/2+i\alpha_n/2, u_{j,l}) \right]; \tag{B.22}$$

the quantities c_l and $u_{j,l}$ have been defined as follows

$$c_l \equiv (r + K_l - i\alpha)^{-1}, \qquad u_{j,l} \equiv (1-j)(r+K_l) + j(i\alpha). \tag{B.23}$$

And furthermore

$$X_8 = -\frac{1}{4K\alpha^3} Im \left\{ \sum_{j,l,m,n=0}^{1} \frac{(-1)^{j+l+m+n} c_l}{3p/2+r+i\alpha_m - i\alpha_n/2} \times [\xi_2(2r+2p+i\alpha_m, u_{j,l}) - \xi_2(r+p/2+i\alpha_n/2, u_{j,l})] \right\}. \tag{B.24}$$

The arguments of the first function ξ_2 can eventually become equal, in which case property A.3 can not be used to reduce it to functions ξ_1; writing down explicitly the terms $j = 0$ and $j = 1$ in the first factor we have

$$X_8 = \frac{1}{4K\alpha^3} Im \left\{ \sum_{m,n=0}^{1} \frac{(-1)^{m+n}}{3p/2+r+i\alpha_m - i\alpha_n/2} \left[(c_o - c_1)\xi_2(2r+2p+i\alpha_m, i\alpha) - \sum_{l=0}^{1} (-1)^l c_l \xi_2(2r+2p+i\alpha_m, r+K_l) \right] + \sum_{j,l,m,n=0}^{1} \frac{(-1)^{j+l+m+n} c_l \xi_2(r+p/2+i\alpha_n/2, u_{j,l})}{3p/2+r+i\alpha_m - i\alpha_n/2} \right\}. \tag{B.25}$$

In the first term the sum over l was separated from the other two sums; the functions ξ_2 that appear in this last result have the following dependence on p (from properties A.3 and A.8)

$$\xi_2(2r+2p+i\alpha_o, i\alpha) = \delta_{p,-r} \xi_2(i\alpha)$$

$$\begin{aligned}
\xi_2(2r+2p+i\alpha_1, i\alpha) &= -\frac{\xi_1(i\alpha)}{r+p-i\alpha} \\
\xi_2(2r+2p+i\alpha_m, r+K_l) &= \frac{(-1)^m\xi_1(i\alpha)-(-1)^l\xi_1(\epsilon+K)}{2p+r-K_l+i\alpha_m} \\
\xi_2(r+p/2+i\alpha_n/2, u_{j,l}) &= \frac{\xi_1(\epsilon+p/2+i\alpha_n/2)-\xi_1(u_{j,l})}{p/2+r-u_{j,l}+i\alpha_n/2};
\end{aligned}$$

we have used the facts that r is an integer plus ϵ, $2r$ is an integer and $\xi_1(\epsilon+A_l)=(-1)^l\xi_1(A)$. Substituting this results in equation B.25 we obtain the result

$$\begin{aligned}
X_8 = \frac{1}{4K\alpha^3} Im \Bigg\{ & (c_o - c_1)\sum_{n=0}^{1}(-1)^n \\
& \left[\frac{\delta_{p,-r}\xi_2(i\alpha)}{-r/2+i\alpha-i\alpha_n/2} + \frac{\xi_1(i\alpha)}{(r+p-i\alpha)(3p/2+r+i\alpha_m-i\alpha_n/2)}\right] \\
& + \sum_{l,m,n=0}^{1} \frac{(-1)^n s_{l,m}}{(2p+r-K_l+i\alpha_m)(3p/2+r+i\alpha_m-i\alpha_n/2)} \\
& + \sum_{j,l,m,n=0}^{1} \frac{(-1)^m v_{j,l,n}(p)}{(p/2+r-u_{j,l}+i\alpha_n/2)(3p/2+r+i\alpha_m-i\alpha_n/2)} \Bigg\},
\end{aligned} \tag{B.26}$$

where the quantities $s_{l,m}$ and $v_{j,l,n}$ have been defined as

$$s_{l,m} \equiv c_l\left[(-1)^m\xi_1(\epsilon+K)-(-1)^l\xi_1(i\alpha)\right], \tag{B.27}$$

$$v_{j,l,n}(p) \equiv (-1)^{j+l+n}c_l\left[\xi_1(\epsilon+p/2+i\alpha_n/2)-\xi_1(u_{j,l})\right]. \tag{B.28}$$

The sum $c_o - c_1$ is equal to $-2Kt$, where t is

$$t \equiv \left[(r-i\alpha)^2 - K^2\right]^{-1}; \tag{B.29}$$

and the sum that multiplies the Kronecker delta is

$$\sum_{n=0}^{1}\frac{(-1)^n}{-r/2+i\alpha-i\alpha_n/2} = \frac{i4\alpha}{(r-i\alpha)(r-i3\alpha)}; \tag{B.30}$$

therefore, the final result for X_8 as a function of p is

$$X_8 = \frac{1}{4K\alpha^3} Im \Bigg\{ -\delta_{p,-r}\frac{i8\alpha K t\xi_2(i\alpha)}{(r-i\alpha)(r-i3\alpha)} \tag{B.31}$$

$$-2Kt\xi_1(i\alpha)\sum_{n=0}^{1}\frac{(-1)^n}{(r+p-i\alpha)(3p/2+r+i\alpha_m-i\alpha_n/2)} \tag{B.32}$$

$$+\sum_{l,m,n=0}^{1}\frac{1}{3p/2+r+i\alpha_m-i\alpha_n/2}\left[\frac{(-1)^n s_{l,m}}{2p+r-K_l+i\alpha_m}+\sum_{j=0}^{1}\frac{(-1)^m v_{j,l,n}(p)}{p/2+r-u_{j,l}+i\alpha_n/2}\right]\Bigg\}.$$

With this result, the sum over p in equation B.16 can now be evaluated; notice that the number $v_{j,l,n}(p/2)$ takes only two different values, depending whether p is even or odd; therefore the sum of the last term in X_8 must be separated into two sums, one over even numbers and the other over odd numbers. The final result is

$$\begin{aligned} N_e^h(r) = -\frac{\beta^4\beta_h^2}{256K\alpha_3} Im \Big\{ & \delta_{\epsilon,0} \frac{i8\alpha t\xi_2(i\alpha)}{K(r-i\alpha)(r-i3\alpha)(r^2+\alpha^2)} \\ & -\frac{4}{3}Kt\xi_1(i\alpha)\sum_{n=0}^{1}\xi_6(\pm i\alpha, r\pm K, r-i\alpha, (2r-i2\alpha-i\alpha_n)/3) \\ & +\frac{1}{3}\sum_{l,m,n=0}^{1}(-1)^n s_{l,m}\,\xi_6(\pm i\alpha, r\pm K, r/2-K_l/2\pm i\alpha_m/2, (2r-i2\alpha_m-i\alpha_n)/3) \\ & +\frac{1}{48}\sum_{j,l,m=0}^{1}(-1)^m\,[v_{j,l,n}(0)\,\xi_6(\pm i\alpha/2, r/2\pm K/2, r-u_{j,l}+i\alpha_n/2, (2r-i2\alpha_m-i\alpha_n)/6) \\ & +v_{j,l,n}(1/2)\,\eta_6(\pm i\alpha/2, r/2\pm K/2, r-u_{j,l}+i\alpha_n/2, (2r-i2\alpha_m-i\alpha_n)/6)]\Big\}. \end{aligned} \tag{B.33}$$

B.3 Exchange diagram in the cluster approximation

To end this appendix we will find an analytic expression for the exchange diagram $\mathcal{D}_e^h$ introduced in chapter 6. The final form given in chapter 6 was

$$\mathcal{D}_e^h(a_1,a_2,a_3,a_4) = -\left(\frac{\beta^2\beta_h}{4}\right)^2 \sum_p X_6(2\epsilon+a_1-a_3,0,a_1)\,X_6(2\epsilon+a_1-a_3,a_2+a_3,0), \tag{B.34}$$

with X_6 defined by

$$X_6(b_1,b_2,b_3) \equiv \xi_6(b_1-p\pm i\alpha, b_2+p\pm i\alpha, b_3+\epsilon\pm K); \tag{B.35}$$

the numbers p, b_1, b_2, and b_3 are all integers, and ϵ can be 0 or 1/2; the Bethe momenta K and α are real and no integers. Using proposition 17, X_6 can be written in terms of functions ξ of the third order

$$X_6 = \frac{1}{2K(i2\alpha)^2}\sum_{l,m,n=0}^{1}(-1)^{l+m+n}\,\xi_3(b_1-p+i\alpha_m, b_2+p+i\alpha_n, b_3+\epsilon+K_l); \tag{B.36}$$

we have used here the same notation of equation B.19, and dropped the arguments of X_6 until we get a final form for it. For every triplet (l,m,n), the term $(l,1-m,1-n)$ gives

its complex conjugate; therefore the sum over n can be written as twice the real part of the term with n equals zero

$$X_6 = -\frac{1}{4K\alpha^2} Re\left[\sum_{l,m=0}^{1} (-1)^{l+m}\, \xi_3(b_1 - p + i\alpha_m, b_2 + p + i\alpha, b_3 + \epsilon + K_l)\right]. \tag{B.37}$$

At this point the algebra to reduce the order of the functions ξ until we get them independent of p follows very closely what we did for X_8 in the previous section; namely, we will apply the results from propositions 6 and 11

$$\begin{aligned} X_6 &= -\frac{1}{4K\alpha^2} Re\left[\sum_{l,m=0}^{1} (-1)^{l+m} \frac{\xi_2(b_1 - p + i\alpha_m, b_2 + p + i\alpha) - \xi_2(b_1 - p + i\alpha_m, b_3 + \epsilon + K_l)}{p + b_2 - b_3 - \epsilon - K_l + i\alpha}\right], \\ &= -\frac{1}{4K\alpha^2} Re\left\{\sum_{l=0}^{1} \frac{(-1)^l}{p + b_2 - b_3 - \epsilon - K_l + i\alpha}\right. \\ &\quad \left.\times\left[\delta(2p + b_2 - b_1)\xi_2(i\alpha) - \frac{2\xi_1(i\alpha)}{2p + b_2 - b_1 + i2\alpha} - \sum_{m=0}^{1} (-1)^m \frac{\xi_1(\epsilon + K_l) - \xi_1(i\alpha_m)}{p + b_3 - b_1 + \epsilon + K_l - i\alpha_m}\right]\right\}. \end{aligned}$$

After a little more algebra we obtain the result

$$\begin{aligned} X_6(b_1, b_2, b_3) = -\frac{1}{4K\alpha^2} Re\left\{\frac{d\,\delta(2p + b_2 - b_1)}{\left(\frac{b_1+b_2}{2} - b_3 - \epsilon + i\alpha\right)^2 - K^2}\right. \\ \left. + \sum_{l=0}^{1}\sum_{m=-1}^{1} \frac{e_{l,m}}{(p - f_l + b_2 - b_3)\left(p - g_{l,m} - \frac{1+|m|}{2} b_1 + \frac{1-|m|}{2} b_2 + |\, m\,|\, b_3\right)}\right\}; \end{aligned} \tag{B.38}$$

the numbers d and $e_{l,m}$, f_l and $g_{l,m}$ have been defined in the following way

$$d \equiv 2K\,\xi_2(i\alpha), \tag{B.39}$$

$$e_{l,m} \equiv m\,\xi_1(\epsilon + K) - (-1)^{l+m}\xi_1(i\alpha), \tag{B.40}$$

$$f_l \equiv \epsilon + K_l - i\alpha, \tag{B.41}$$

$$g_{l,m} \equiv -|\, m\,|\,(\epsilon + K_l) + (|\, m\,| - m - 1)i\alpha; \tag{B.42}$$

d is real and e, f and g are complex.

The last result for X_6 can be used in equation B.34 to evaluate the sum over p

$$\mathcal{D}_e^h(a_1, a_2, a_3, a_4) = -\left(\frac{\beta^2\beta_h}{16K\alpha^2}\right)^2 \sum_p \tag{B.43}$$

$$Re\left[\frac{4d\,\delta(2p-2\epsilon+a_3-a_1)}{(a_3+a_1-i2\alpha)^2-k^2}+\sum_{l=0}^{1}\sum_{m=-1}^{1}\frac{e_{l,m}}{(p-f_l-a_1)(p-g_{l,m}-A_m)}\right]$$
$$\times Re\left[\frac{4d\,\delta(2p-2\epsilon+2a_3+a_2-a_1)}{(a_2+a_1+i2\alpha)^2-k^2}+\sum_{j=0}^{1}\sum_{n=-1}^{1}\frac{e_{j,n}}{(p-f_j+a_2+a_3)(p-g_{j,n}-B_n)}\right];$$

the numbers A_m and B_n are given by

$$A_o=\frac{a_1-a_3}{2}, \qquad A_{\pm1}=-a_3,$$
$$B_o=\frac{a_1-a_2}{2}-a_3, \qquad B_{\pm1}=a_1-a_3. \tag{B.44}$$

The use of the relation $Re(z_1)\,Re(z_2)=Re[z_1\,Re(z_2)]$ leads to

$$\mathcal{D}_e^h(a_1,a_2,a_3,a_4)=-\left(\frac{\beta^2\beta_h}{4K\alpha^2}\right)^2\sum_p\left\{\frac{1}{16}d^2R_1R_2\,\delta_{a_2,-a_3}\delta(p-\epsilon-A_o)\right. \tag{B.45}$$
$$+\frac{1}{4}R_1d\,\delta(p-\epsilon-A_o)Re\left[\sum_{j=0}^{1}\sum_{n=-1}^{1}\frac{e_{j,n}}{2(2\epsilon-2f_j+a_2-a_1)(\epsilon-g_{j,n}+A_o-B_m)}\right]$$
$$+\frac{1}{4}R_2d\,\delta(p-\epsilon-B_o)Re\left[\sum_{j=0}^{1}\sum_{n=-1}^{1}\frac{e_{j,n}}{2(2\epsilon-2f_j+a_1-a_3)(\epsilon-g_{j,n}+B_o-A_m)}\right]$$
$$+\frac{1}{32}Re\sum_{j,l=0}^{1}\sum_{m,n=-1}^{1}e_{l,m}\left[\frac{e_{j,n}}{(p-f_l-a_1)(p-g_{l,m}-A_m)(p-f_j+a_2+a_3)(p-g_{j,n}-B_n)}\right.$$
$$\left.\left.+\frac{e^\star_{j,n}}{(p-f_l-a_1)(p-g_{l,m}-A_m)(p-f^\star_j+a_2+a_3)(p-g^\star_{j,n}-B_n)}\right]\right\}.$$

Here the real numbers R_1 and R_2 have been defined as

$$R_1\equiv Re\left[\frac{1}{(A_o-A_1-i2\alpha)^2-K^2}\right], \tag{B.46}$$

$$R_2\equiv Re\left[\frac{1}{(B_o-B_1-i2\alpha)^2-K^2}\right], \tag{B.47}$$

The sum over p can now be easily evaluated, and the final result for $\mathcal{D}_e^h(a_1,a_2,a_3,a_4)$ is

$$\mathcal{D}_e^h(a_1,a_2,a_3,a_4)=-\frac{1}{16}\left(\frac{\beta}{\alpha}\right)^4\left(\frac{\beta_h}{4K}\right)^2\left\{d^2R_1R_2\,\delta_{a_2,-a_3}\Delta(2\epsilon+a_1-a_3)\right. \tag{B.48}$$
$$+R_1d\,\Delta(2\epsilon+a_1-a_3)Re\left[\sum_{j=0}^{1}\sum_{n=-1}^{1}\frac{e_{j,n}}{(\epsilon-f_j+a_2/2-a_1/2)(\epsilon-g_{j,n}+A_o-B_m)}\right]$$
$$+R_2d\,\Delta(2\epsilon+a_1-a_2)Re\left[\sum_{j=0}^{1}\sum_{n=-1}^{1}\frac{e_{j,n}}{(\epsilon-f_j+a_1/2-a_3/2)(\epsilon-g_{j,n}+B_o-A_m)}\right]$$

$$+\frac{1}{2}Re\sum_{j,l=0}^{1}\sum_{m,n=-1}^{1} e_{l,m}\left[e_{j,n}\,\xi_4(f_l+a_1, g_{l,m}+A_m, f_j-a_2-a_3, g_{j,n}+B_n)\right.$$
$$\left.+e^*_{j,n}\,\xi_4(f_l+a_1, g_{l,m}+A_m, f^*_j-a_2-a_3, g^*_{j,n}+B_n)\right]\Big\};$$

the function Δ is the simple function

$$\Delta(n) \equiv (m+1) \bmod 2. \tag{B.49}$$

This result may seem much more complicated than the exchange diagram in the quark model, $\mathcal{D}^h_e(a_1, a_2, a_3, a_4)$, which was written in chapter 6 in a more compact form. However, the result in the quark model involves two sums over the symmetric group S_4 and the other over half of the elements of S_4, for a total of 288 terms, in contrast to the few terms obtained in the cluster approximation.

Appendix C

The Bethe ansatz for four fermions.

In this appendix we use the result given by Kebukawa [22] for the momentum representation of the Bethe *ansatz* of fermions with two colors, and show how to rewrite it in the simple form of section 4.3.2; we will consider only the particular case of four particles, two of each color. In that case the subindex j in the *ansatz* equations (4.59) and (4.60) takes values from 1 to 4; the superindex α takes only two values, which we will label as -1 and 1. Corresponding to $\alpha = \pm 1$ there are two momenta Λ^α which we will simply label Λ^+ and Λ^-.

Since $\omega^{\alpha,\beta}$ is antisymmetric, it has only one independent component, which according to (4.62) is

$$\omega \equiv \frac{1}{\pi}\cot^{-1}\left[\frac{2\pi(\Lambda^+ - \Lambda^-)}{cL}\right]; \tag{C.1}$$

we are using here dimensionless units in which momenta are given in units of $2\pi/L$. The auxiliary momenta k_j^α introduced in equation (4.61) become

$$k_j^+ = \frac{1}{\pi}\cot^{-1}\left[\frac{4\pi(K_j - \Lambda^+)}{cL}\right] \quad (j = 1,2,3,4), \tag{C.2}$$

$$k_j^- = \frac{1}{\pi}\cot^{-1}\left[\frac{4\pi(K_j - \Lambda^-)}{cL}\right] \quad (j = 1,2,3,4). \tag{C.3}$$

The Bethe momenta K_j and Λ^α must be the solution to the system of equations (4.59)

and (4.60)

$$K_j = n_j + k_j^+ + k_j^- \quad (j = 1,2,3,4), \tag{C.4}$$

$$\lambda^\alpha = k_1^\alpha + k_2^\alpha + k_3^\alpha + k_4^\alpha + \alpha\omega \quad (\alpha = -1, 1); \tag{C.5}$$

the quantum numbers $\{n_1, n_2, n_3, n_4, \lambda^+, \lambda^-\}$ are all integers.

From the general result given by Kebukawa [22] we have

$$\begin{aligned} |\psi\rangle = & \sum_{1\le\nu_1<\nu_2\le 4} \sum_{p_1^+,\ldots,p_4^+} \sum_{p_1^-,\ldots,p_4^-} \sum_R \frac{\Delta_{1,2}}{R-\omega} \\ & \times [f_{\nu_3}^+ f_{\nu_3}^- f_{\nu_4}^+ f_{\nu_4}^- (f_{\nu_1}^+ f_{\nu_2}^- + f_{\nu_1}^- f_{\nu_2}^+)]\, O^\dagger(p_1,p_2,p_3,p_4;\nu_1,\nu_2)\,|0\rangle. \end{aligned} \tag{C.6}$$

where the following definitions have been made

$$\Delta_{1,2} \equiv \delta\left(\sum_{j=1}^4 p_j^+ + R - \lambda^+\right)\ \delta\left(\sum_{j=1}^4 p_j^- - R - \lambda^-\right), \tag{C.7}$$

$$f_j^\alpha \equiv (p_j^\alpha - k_j^\alpha)^{-1}, \tag{C.8}$$

$$p_j \equiv n_j + p_j^+ + p_j^-. \tag{C.9}$$

The operator $O^\dagger(p_1,p_2,p_3,p_4;l_1,l_2)$ stands for the product of four creation operators with indices p_1, p_2, p_3 and p_4, of which the ones in the positions ν_1 and ν_2 create particles of color a, and the rest create particles of color b.

We will prove in this appendix that these eigenstates can be written in a simpler form that involves only the Bethe momenta K_i instead of the auxiliary momenta k_i^α. Since the creation operator $O^\dagger$ does not depend on p_j^+ and p_j^- independently, we will make a change of dummy indices in such a way that the field operators can be taken out of some of the sums; let us define the variables

$$r_j \equiv \left[\frac{p_j^+ - p_j^-}{2}\right]^-, \tag{C.10}$$

where $[A]^-$ stands for the largest integer that is less or equal to A. With this definition and the p_j 's defined above, we can replace the variables p_j^+ and p_j^- by

$$p_j^+ = \left[\frac{p_j - n_j}{2}\right]^+ + r_j, \tag{C.11}$$

$$p_j^- = \left[\frac{p_j - n_j}{2}\right]^- - r_j; \tag{C.12}$$

and the state of equation (C.6) can be written in the form

$$|\psi\rangle = \sum_{\nu\in S_4} \sum_{p_1,\ldots,p_4} \sum_{r_o,\ldots,r_4} \frac{\Delta_{1,2}}{r_o-\omega} \left[f^+_{\nu_3} f^-_{\nu_3} f^+_{\nu_4} f^-_{\nu_4} (f^+_{\nu_1} f^-_{\nu_2} + f^-_{\nu_1} f^+_{\nu_2})\right] O^\dagger(p_1,p_2,p_3,p_4;\nu_1,\nu_2)\,|0\rangle; \tag{C.13}$$

here we have used the fact that the indices ν_1 and ν_2 can be exchanged and the result is the same state; the indices ν_3 and ν_4 are the positions where there are operators with color b in $O^\dagger$. In going from the sum over distinct pairs (ν_1,ν_2) to the sum over the symmetric group S_4, we should introduced a factor of $1/4$, which has not been written since it only changes the norm of the state/

The functions f^α_j take the following forms as functions of p_j and r_j

$$f^\alpha_j = \alpha\left(r_j - \Gamma^\alpha_j\right)^{-1}, \tag{C.14}$$

where

$$\Gamma^\alpha_j \equiv \alpha\left(k^\alpha_j - \left[\frac{p_j-n_j}{2}\right]^\alpha\right). \tag{C.15}$$

We now notice that

$$\begin{aligned}
\sum_{j=1}^4 \left(p^+_j + p^-_j\right) &= \sum_{j=1}^4 \left(\left[\frac{p_j-n_j}{2}\right]^+ + \left[\frac{p_j-n_j}{2}\right]^-\right), \\
&= \sum_{j=1}^4 p_j + \lambda^+ + \lambda^- - Q, \\
\sum_{j=1}^4 \left(p^+_j - p^-_j\right) &= \sum_{j=1}^4 \left(\left[\frac{p_j-n_j}{2}\right]^+ + \left[\frac{p_j-n_j}{2}\right]^-\right) + 2\,r_j, \\
&= \sum_{j=1}^4 \left(\epsilon_j + 2\,r_j\right),
\end{aligned}$$

where the integers ϵ_j can be either 0 or 1, depending on the parity of $p_j - n_j$

$$\epsilon_j \equiv (p_j - n_j) \bmod 2, \tag{C.16}$$

and the integer Q is the sum of all the quantum numbers

$$Q \equiv \lambda^+ + \lambda^- + \sum_{j=1}^4 n_j. \tag{C.17}$$

Thus, the function $\Delta_{1,2}$ of equation (C.7), becomes

$$\Delta_{1,2} = \delta(p_1 + p_2 + p_3 + p_4 - Q)\, \delta\left(\sum_{j=1}^{4}(2\,p_j + \epsilon_j) + 2\,R + \lambda^- - \lambda^+\right). \tag{C.18}$$

The operator $O^\dagger$ does not depend on the indices r_j; therefore, the sums over r_j can be evaluated over the functions f_j^α, and the state of equation (C.6) becomes

$$|\,\psi\rangle = \sum_{p_1,\ldots,p_4} \delta(p_1+p_2+p_3+p_4-Q) \sum_{\nu\in S_4} \Omega_\nu(p_1,\ldots,p_4)\; O^\dagger(p_1,\ldots,p_4;\nu_1,\nu_2)\,|\,0\rangle. \tag{C.19}$$

Here the wave function Ω_ν is given by,

$$\Omega_\nu \equiv \sum_{r_o,\ldots,r_4} \delta(r_o+r_1+r_2+r_3+r_4-M)\frac{f^+_{\nu_3} f^-_{\nu_3} f^+_{\nu_4} f^-_{\nu_4}(f^+_{\nu_1} f^-_{\nu_2} + f^-_{\nu_1} f^+_{\nu_2})}{r_o - \omega}, \tag{C.20}$$

$$M \equiv \frac{1}{2}(\lambda^+ + \lambda^- - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4); \tag{C.21}$$

from equation (C.17), and since $Q = p_1 + \cdots + p_4$, it follows that M is an integer.

The sums over the indices r_j can now be evaluated. First notice that from the definition of the numbers Γ_j^α it follows that

$$\Gamma_j^+ - \Gamma_j^- = p_j - K_j, \tag{C.22}$$

which will never vanish, because K_j is not an integer; then the product $f_j^+ f_j^-$ can be written as

$$\begin{aligned} f_j^+ f_j^- &= -(r_j - \Gamma_j^+)^{-1}(r_j - \Gamma_j^-)^{-1} \\ &= (\Gamma_j^+ - \Gamma_j^-)^{-1}\left[(r_j - \Gamma_j^+)^{-1} - (r_j - \Gamma_j^-)^{-1}\right] \\ &= \frac{f_j^+ + f_j^-}{p_j - K_j}. \end{aligned} \tag{C.23}$$

With this relation the wave function becomes

$$\begin{aligned} \Omega_\nu(p_1,\ldots,p_4) &= -(p_{\nu_3} - K_{\nu_3})^{-1}(p_{\nu_4} - K_{\nu_4})^{-1} \\ &\times \widetilde{\sum_{\alpha_1,\ldots,\alpha_4}}\, \alpha_3\alpha_4 \sum_{r_o,\ldots,r_4} \delta(r_o + \cdots + r_4)(r_o + M - \omega)^{-1}(r_1 - \Gamma_{\nu_1}^{\alpha_1})^{-1}\cdots(r_4 - \Gamma_{\nu_4}^{\alpha_4})^{-1}; \end{aligned} \tag{C.24}$$

the sum over the indices $\alpha_1, \ldots, \alpha_4$ runs over values of 1 or -1 for each index, but the indices α_1 and α_2 cannot be equal. The last sum has the form of the function S_1 defined in appendix B; using the result found there, the wave function takes the form [1]

$$\Omega_\nu(p_1,p_2,p_3,p_4) = \frac{\pi^4}{(p_{\nu_3} - K_{\nu_3})(p_{\nu_4} - K_{\nu_4})} \widetilde{\sum_{\alpha_1,\ldots,\alpha_4}} \frac{G(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}{\omega - M + \Gamma_{\nu_1}^{\alpha_1} + \cdots + \Gamma_{\nu_4}^{\alpha_4}}, \tag{C.25}$$

where the function G is defined in the following way

$$G(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \equiv \frac{\alpha_3\alpha_4 \sin\left[\pi(\omega + \Gamma_{\nu_1}^{\alpha_1} + \cdots + \Gamma_{\nu_4}^{\alpha_4})\right]}{\sin(\pi\omega)\sin(\pi\Gamma_{\nu_1}^{\alpha_1})\cdots\sin(\pi\Gamma_{\nu_4}^{\alpha_4})}. \tag{C.26}$$

From the definitions of M and Γ_j^α, and using equation (C.5) we have

$$\begin{aligned} \omega - M + \Gamma_{\nu_1}^{\alpha_1} + \cdots + \Gamma_{\nu_4}^{\alpha_4} &= \sum_{j=1}^{4} \left\{ \frac{1}{2}(k_j^- - k_j^+) + \frac{1}{2}\left[\frac{p_j - n_j}{2}\right]^+ - \frac{1}{2}\left[\frac{p_j - n_j}{2}\right]^- \right. \\ &\qquad \left. -\alpha_j \left[\frac{p_{\nu_j} - n_{\nu_j}}{2}\right]^{\alpha_j} + \alpha_j k_{\nu_j} \right\} \\ &= -\frac{1}{2}\sum_{j=1}^{4} \alpha_j (p_{\nu_j} - K_{\nu_j}); \end{aligned} \tag{C.27}$$

furthermore, the two Bethe *ansatz* equations (C.4) and (C.5), and the definition of Q imply

$$K_1 + K_2 + K_3 + K_4 = Q; \tag{C.28}$$

and since the sum of p_1, p_2, p_3 and p_4 is also equal to Q, the expression above can be written as

$$\omega - M + \Gamma_{\nu_1}^{\alpha_1} + \cdots + \Gamma_{\nu_4}^{\alpha_4} = -\frac{1}{2}\sum_{j=1}^{4}(\alpha_j + 1)(p_{\nu_j} - K_{\nu_j}) = -\frac{1}{2}\sum_{j=1}^{4}(\alpha_j - 1)(p_{\nu_j} - K_{\nu_j}); \tag{C.29}$$

therefore, the sum over j on the right can be restricted only to those values of j for which the corresponding indices α_j are 1; or equivalently, the sum can be done only over indices j with $\alpha_j = -1$.

Going back to the wave function Ω_ν, the sums over indices α_j lead to eight terms; four of them have half of the indices α_j equal to 1, and the other half equal to -1; the other form

[1] For the time being, we assume that $\omega + \Gamma_{\nu_1}^{\alpha_1} + \cdots + \Gamma_{\nu_4}^{\alpha_4}$ is not equal to zero; at the end of the chapter we will consider what would happen when this assumption is not true.

terms have three identical indices α_j and the other one different. The eight terms can be arranged in the following form

$$\Omega_\nu(p_1,p_2,p_3,p_4) = \frac{\pi^4}{(p_{\nu_3}-K_{\nu_3})(p_{\nu_4}-K_{\nu_4})} \tag{C.30}$$
$$\times\left[\frac{G^-_{\nu_1}-G^+_{\nu_1}}{p_{\nu_1}-K_{\nu_1}}+\frac{G^-_{\nu_2}-G^+_{\nu_2}}{p_{\nu_2}-K_{\nu_2}}+\frac{G^+_{\nu_1,\nu_3}-G^-_{\nu_1,\nu_3}}{p_{\nu_1}+p_{\nu_3}-K_{\nu_1}-K_{\nu_3}}+\frac{G^+_{\nu_1,\nu_4}-G^-_{\nu_1,\nu_4}}{p_{\nu_1}+p_{\nu_4}-K_{\nu_1}-K_{\nu_4}}\right];$$

here the symbol $G^+_{j\ldots l}$ stands for $G(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ with $\alpha_j = 1$ only if j is one of the subindices of $G^+_{j\ldots l}$; and the opposite for $G^-_{j\ldots l}$. For example

$$G^-_{1,3} = G(-1,1,-1,1) = G^+_{2,4}.$$

Now that we have the wave function in a simple analytic form, we return now the simplification of the state $|\psi\rangle$. Using the anticommutation relations of $a^\dagger$ and $b^\dagger$, we have

$$O^\dagger(p_1,p_2,p_3,p_4;\nu_1,\nu_2) = (-1)^\nu\, a^\dagger_{p_{\nu_1}} a^\dagger_{p_{\nu_2}} b^\dagger_{p_{\nu_3}} b^\dagger_{p_{\nu_4}}, \tag{C.31}$$

where $(-1)^\nu$ is the parity of the permutation ν . Introducing this last equation into equation (C.19), and since the integers p_j are dummy indices, we can write the state in the form

$$|\psi\rangle = \sum_{p_1,\ldots,p_4} \delta(p_1+p_2+p_3+p_4-Q)\Omega(p_1,p_2,p_3,p_4)\, a^\dagger_{p_1} a^\dagger_{p_2} b^\dagger_{p_3} b^\dagger_{p_4}\, |0\rangle, \tag{C.32}$$

where the wave function $\Omega(p_1,p_2,p_3,p_4)$ has been defined as

$$\Omega(p_1,p_2,p_3,p_4) = \sum_{\nu\in S_4} \frac{(-1)^\nu\pi^4}{(p_3-K_{\nu_3})(p_4-K_{\nu_4})} \tag{C.33}$$
$$\times\left[\frac{G^+_{\nu_1}-G^-_{\nu_1}}{p_1-K_{\nu_1}}+\frac{G^+_{\nu_2}-G^-_{\nu_2}}{p_2-K_{\nu_2}}+\frac{G^+_{\nu_1,\nu_3}-G^-_{\nu_1,\nu_3}}{p_1+p_3-K_{\nu_1}-K_{\nu_3}}+\frac{G^+_{\nu_1,\nu_4}-G^-_{\nu_1,\nu_4}}{p_1+p_4-K_{\nu_1}-K_{\nu_4}}\right];$$

this function is antisymmetric under the excahnge of p_1 with p_2 and p_3 with p_4. However, the wave function does not have to be antisymmetric; for example the following choice of wave function leads to the same state

$$\Omega(p_1,p_2,p_3,p_4) = \sum_{\nu\in S_4} \frac{(-1)^\nu\pi^4}{(p_3-K_{\nu_3})(p_4-K_{\nu_4})}\left[\frac{G^+_{\nu_1}-G^-_{\nu_1}}{p_1-K_{\nu_1}}+\frac{G^+_{\nu_1,\nu_3}-G^-_{\nu_1,\nu_3}}{p_1+p_3-K_{\nu_1}-K_{\nu_3}}\right]. \tag{C.34}$$

The functions G^α_j can be written in a simpler form. Equation (C.26) can be written in terms of cotangents only; then, From equations (C.1), (C.2) and (C.3) we obtain the following

$$\cot(\pi k^+_j) - \cot(\pi k^-_j) = -2\cot(\pi\omega); \tag{C.35}$$

this relation, and the identity

$$\cot(\pi\Gamma_j^\alpha) = \alpha\ \cot(\pi k_j^\alpha), \tag{C.36}$$

allow us to write (after some algebra) the two numerators of the function Ω in the form

$$\begin{aligned} G_j^+ - G_j^- &= 4[\cot^3(\pi\omega) + \cot(\pi\omega)][2\cot(\pi\omega) + \cot(\pi k_1^+) + \cdots + \cot(\pi k_4^+) - 2\cot(\pi k_j^+)], \\ G_{ij}^+ - G_{ij}^- &= 4[\cot^3(\pi\omega) + \cot(\pi\omega)] \\ &\quad \times[2\cot(\pi\omega) + \cot(\pi k_1^+) + \cdots + \cot(\pi k_4^+) - 2\cot(\pi k_i^+) - 2\cot(\pi k_j^+)]; \end{aligned}$$

the common factor inside the first square brackets can be ignored, because it only changes the norm of the state; and if the auxiliary momenta are replaced by the Bethe momenta, using equations (C.1), (C.2) and (C.3) we have

$$G_j^+ - G_j^- = \frac{4\pi}{cL}(Q - 2K_j - \Lambda^+ - \Lambda^-), \tag{C.37}$$

$$G_{ij}^+ - G_{ij}^- = \frac{4\pi}{cL}(2K_i + 2K_j - Q). \tag{C.38}$$

The wave function thus can be written as

$$\Omega(p_1,p_2,p_3,p_4) = \sum_{\nu\in S_4} \frac{(-1)^\nu \pi^4}{(p_3 - K_{\nu_3})(p_4 - K_{\nu_4})} \left[\frac{Q - 2K_{\nu_1} - \Lambda^+ - \Lambda^-}{p_1 - K_{\nu_1}} + \frac{2K_{\nu_1} + 2K_{\nu_3} - Q}{p_1 + p_3 - K_{\nu_1} - K_{\nu_3}}\right];$$

finally, the translational invariance of the state can shown more explicitly if we redefine the Bethe momenta $\{K_1, K_2, K_3, K_4, \Lambda^+, \Lambda^-\}$ by substracting $Q/4$ to all of them. The resulting momenta are solutions to the two equations (4.68) and (4.69) given in page 53; the state can then be written in the form

$$|\psi\rangle = \frac{1}{\sqrt{4}} \sum_{p_1,\ldots,p_4} \Omega(p_1 - Q/4, \ldots, p_4 - Q/4)\ a_{p_1}^\dagger a_{p_2}^\dagger b_{p_3}^\dagger b_{p_4}^\dagger\ |0\rangle, \tag{C.39}$$

where the wave function Ω is

$$\Omega(p_1,p_2,p_3,p_4) = \sum_{\nu\in S_4} \frac{(-1)^\nu \delta(p_1+p_2+p_3+p_4)}{(p_3 - K_{\nu_3})(p_4 - K_{\nu_4})} \left[\frac{K_{\nu_1} + (\Lambda^+ + \Lambda^-)/2}{p_1 - K_{\nu_1}} - \delta_\nu \frac{K_{\nu_1} + K_{\nu_3}}{p_1 + p_3 - K_{\nu_1} - K_{\nu_3}}\right]; \tag{C.40}$$

and the Bethe momenta are given by the solution to equations (4.68) and (4.69). The number δ_ν means the following: the sum of two Bethe momenta can become zero, in which case the

algebra that we have used to derive the wave function is not valid; in that case, a similar procedure to the one we followed leads to the cancelation of the last term in Ω. This is indicated by the number δ_ν; if the permutation ν leads to $K_{\nu_1} + K_{\nu_3} = 0$, then the last fraction must be ignored; if the sum is not zero, then the number δ_ν is given a value of 1.

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